**General Mathematics Vol. 13, No. 1 (2005), 83–90**

# First order nonlinear differential superordination

**Georgia Irina Oros**

*Dedicated to Professor Emil C. Popa on his 60th anniversary*

**Abstract**

In this paper we shall extend the first-order linear differential superordination defined by the authors in [2] to a first-order nonlinear differential superordination of the form:

$$U \subset \{\lambda(z)zp'(z) + \mu(z)p^2(z) + p(z);\ z \in U\}.$$

**2000 Mathematics Subject Classification:** Primary 30C80, Secondary 30C45, 30A20, 34A40.

**Keywords:** differential subordination, differential superordination, univalent.

## 1 Introduction

Let $\Omega$ be any set in the complex plane $\mathbb{C}$, let $p$ be analytic in the unit disk $U$ and let $\psi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. In a series of articles the authors and many others [1] have determined properties of functions $p$ that satisfy the differential subordination

$$\{\psi(p(z), zp'(z), z^2p'(z); z)|\ z \in U\} \subset \Omega.$$

In this article we consider the dual problem of determining properties of functions $p$ that satisfy the differential superordination

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p'(z); z)|\ z \in U\}.$$

This problem was introduced in [2].

We let $\mathcal{H}(U)$ denote the class of holomorphic functions in the unit disk $U = \{z \in \mathbb{C} :\ |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U),\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots,\ z \in U\}.$$

For $0 < r < 1$, we let $U_r = \{z \in \mathbb{C},\ |z| < r\}$.

**Definition 1.** [2] *Let* $\varphi : \mathbb{C}^2 \times U \to \mathbb{C}$ *and let* $h$ *be analytic in* $U$*. If* $p$ *and* $\varphi(p(z), zp'(z); z)$ *are univalent in* $U$ *and satisfy the (first-order) differential superordination*

$$h(z) \prec \varphi(p(z), zp'(z); z) \tag{1}$$

*then* $p$ *is called a solution of the differential superordination. An analytic function* $q$ *is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if* $q \prec p$ *for all* $p$ *satisfying (1). A univalent subordinant* $\widetilde{q}$ *that satisfies* $q \prec \widetilde{q}$ *for all subordinants* $q$ *of (1) is said to be the best subordinant. Note that the best subordinant is unique up to a rotation of* $U$*.*

For $\Omega$ a set in $\mathbb{C}$, with $\varphi$ and $p$ as given in Definition 1, suppose (1) is replaced by

$$(1') \qquad \Omega \subset \{\varphi(p(z), zp'(z); z) |\ z \in U\}.$$

Although this more general situation is a "differential containment", the condition in (1 will also be referred to as a differential superordination, and the definitions of solution, subordinant and best dominant as given above can be extended to this generalization.

**Definition 2.** [2] *We denote by $Q$ the set of functions $f$ that are analytic and injective on $\overline{U} \setminus E(f)$, where*

$$E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$$

*and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.*

*The subclass of $Q$ for which $f(0) = a$ is denoted by $Q(a)$.*

**Definition 3.** [2] *Let $Q$ be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\phi_n[\Omega, q]$, consists of those functions $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$(2) \qquad \varphi\left(q(z), \frac{zq'(z)}{m}; \zeta\right) \in \Omega$$

*where $z \in U$, $\zeta \in \partial U$ and $m \geq n \geq 1$.*

In order to prove the new results we shall use the following lemma:

**Lemma A.** [2] *Let $\Omega \subset \mathbb{C}$, $q \in \mathcal{H}[a, n]$, $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$, and suppose that*

$$(3) \qquad \varphi(q(z), tzq'(z); \zeta) \in \Omega,$$

*for $z \in U$, $\zeta \in \partial U$ and $0 < t \leq \dfrac{1}{n} \leq 1$. If $p \in Q(a)$ and $\varphi(p(z), zp'(z); z)$ is univalent in $U$, then*

$$\Omega \subset \{\varphi(p(z), zp'(z); z) |\ z \in U\} \text{ implies } q(z) \prec p(z).$$

**Lemma B.** [1 , Lemma 2.2.d p. 24] *Let* $q \in Q$, *with* $q(0) = a$, *and let*

$$p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$$

*be analytic in* $U$ *with* $p(z) \not\equiv a$ *and* $n \geq 1$. *If* $p$ *is not subordinate to* $q$, *then there exist points* $z_0 = r_0 e^{i\theta_0} \in U$, $r_0 < 1$ *and* $\zeta_0 \in \partial U \setminus E(q)$, *and an* $m \geq n \geq 1$ *for which* $p(U_{r_0}) \subset q(U)$,

*(i)* $p(z_0) = q(\zeta_0)$

*(ii)* $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$, *and*

*(iii)* $\mathrm{Re}\, \dfrac{z_0 p'(z_0)}{p'(z_0)} + 1 \geq m \mathrm{Re}\, \left[\dfrac{\zeta_0 q'(\zeta_0)}{q'(\zeta_0)} + 1\right]$.

# 2 Main results

**Theorem 1.** *Let* $\lambda, \mu : \overline{U} \to \mathbb{C}$ *with* $|\lambda(\zeta)| \leq 1$, $|\mu(\zeta)| \leq 1$, $\zeta = e^{i\theta}$, $p \in \mathcal{H}[0,1] \cap Q$ *and let*

$$\lambda(z) z p'(z) + \mu(z) p^2(z) + p(z)$$

*be univalent in unit disk* $U$.

*If*

$$U \subset \{\lambda(z) z p'(z) + \mu(z) p^2(z) + p(z);\ z \in U\}$$

*or*

$$z \prec \lambda(z) z p'(z) + \mu(z) p^2(z) + p(z)$$

*then*

$$U_r \subset p(U) \text{ or } rz \prec p(z),$$

*where $r$ is given by*

$$r=\sqrt{2}-1. \tag{4}$$

**Proof.** Let $\Omega=\{w\in\mathbb{C}|\ |w|<1\}=U$ and let $q(z)=rz$, $q(U)=U_r=\{w\in\mathbb{C}|\ |w|<r\}=\Delta$.

We let

$$\varphi(p(z),zp'(z);z|\ z\in U)=\lambda(z)zp'(z)+\mu(z)p^2(z)+p(z).$$

In order to apply Lemma A to prove this result we only need to show that admissibility condition holds

$$|\varphi(q(z),tzq'(z);\zeta)|=|\lambda(\zeta)tzq'(z)+\mu(\zeta)q^2(z)+q(z)|$$

$$=|\lambda(\zeta)trz+\mu(\zeta)r^2z^2+rz|=|z|r|\lambda(\zeta)t+\mu(\zeta)rz+1|$$

$$\leq r[|\lambda(\zeta)t+1|+r|z|\cdot|\mu(\zeta)]=r^2+2r\leq 1.$$

Since $|\varphi(q(z),tzq'(z);\zeta)|\in U$, by using Lemma A it results $U_r\subset p(U)$, or $rz\prec p(z)$.

**Remark 1.** *If $\mu(z)\equiv 0$, we obtain the result from [2, Theorem 10].* If $\lambda(z)=-z$, $\mu(z)\equiv 1$ then the differential equation

$$-z\cdot zq'(z)+q^2(z)+q(z)=z$$

has the univalent solution $q(z)=z$. Hence from the sharp form of Theorem 1 we obtain the following result.

**Corollary 1.** *If $p\in\mathcal{H}[0,1]\cap Q$ and $-z^2p'(z)+p^2(z)+p(z)$ is univalent, then $z\prec -z^2p'(z)+p^2(z)+p(z)$, implies*

$$z\prec p(z),\quad z\in U.$$

*The function $z$ is the best subordinant.*

**Remark 2.** *If $\lambda(z) \equiv 1$, $\mu(z) \equiv 0$, we obtain the result from [2, Corollary 10.1].*

If $\lambda(z) = \dfrac{-z}{4}$, $\mu(z) = \dfrac{1}{4}$, then the differential equation

$$-\frac{z}{4} z q'(z) + \frac{1}{4} q^2(z) + q(z) = z$$

has the univalent solution $q(z) = z$. Hence from the sharp form of Theorem 1 we obtain the following result.

**Corollary 2.** *If $p \in \mathcal{H}[0,1] \cap Q$ and*

$$-\frac{z}{4} z p'(z) + \frac{1}{4} p^2(z) + p(z)$$

*is univalent, then*

$$z \prec -\frac{z^2}{4} p'(z) + \frac{1}{4} p^2(z) + p(z)$$

*implies*

$$z \prec p(z), \quad z \in U.$$

*The function $z$ is the best subordinant.*

**Theorem 2.** *Let $N > 1$, $M > 0$, $\lambda, \mu : \overline{U} \to \mathbb{C}$, with $|\lambda(z)|^2 + \mathrm{Re}\ \lambda(z) \geq 0$, $p \in \mathcal{H}[0,1] \cap Q$, and*

$$M[|\lambda(z) + 1| - M|\mu(z)|] \geq N.$$

*If*

$$\lambda(z) z p'(z) + \mu(z) p^2(z) + p(z) \prec Nz$$

*then*

$$p(z) \prec Mz.$$

**Proof.** If we let

$$w(z) = \lambda(z)zp'(z) + \mu(z)p^2(z) + p(z),$$

then

$$|w(z)| < N. \tag{5}$$

Let $q(z) = Mz$. If $p(z) \not\prec q(z)$, then by Lemma B there exist $z_0 \in U$, $\zeta \in \partial U$ and $m > 1$ such that $p(z_0) = q(\zeta) = M\zeta$ and $z_0p'(z_0) = m\zeta q'(\zeta) = mM\zeta$.

For $z_0 \in U$ we have

$$E = |w(z_0)| = |\lambda(z_0)z_0p'(z_0) + \mu(z_0)p^2(z_0) + p(z_0)| =$$

$$= |\lambda(z_0)mNM\zeta + \mu(z_0)M^2\zeta + M\zeta| =$$

$$= M|\lambda(z_0)m + \mu(z_0)M\zeta + 1| \geq M[|\lambda(z_0)m + 1| - M|\mu(z_0)|].$$

From $|\lambda(z)|^2 + \mathrm{Re}\,\lambda(z) \geq 0$ we have

$$|\lambda(z_0)m + 1| > |\lambda(z_0) + 1|.$$

Hence

$$E \geq M[|\lambda(z_0) + 1| - M|\mu(z_0)|] \geq N.$$

Since this contradicts (5), we obtain the desired result $p(z) \prec Mz$.

Under the conditions of Theorem 1 and Theorem 2 we have the following sandwich type result.

**Corollary 3.** *If*

$$z \prec \lambda(z)zp'(z) + \mu(z)p^2(z) + p(z) \prec Nz$$

*then*

$$rz \prec p(z) \prec Mz,$$

*where* $r$ *is given by (4) and* $M > 0$.

## References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

[2] S. S. Miller and P. T. Mocanu, *Subordinants of Differential Superordinations*, Complex Variables, vol. 48, no. 10, 815-826.

Department of Mathematics
University of Oradea
Str. Armatei Române, No. 5
410087 Oradea, Romania
e-mail: georgia_oros_ro@yahoo.co.uk