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A New Criterion for Meromorphically Convex Functions of Order α

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

Let $J_n(\alpha)$ denote the classes of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k,$$

which are regular in the punctured disc $U^* = \{z : 0 < |z| < 1\}$ and satisfy

$$\operatorname{Re}\left\{\frac{(D^{n+1}f(z))'}{(D^n f(z))'} - 2\right\} < -\frac{n+\alpha}{n+1}, \quad z \in U^*,$$

$n \in \mathbb{N} = \{0, 1, ...\}$ and $\alpha \in [0, 1)$, where

$$D^n f(z) = \frac{1}{z}\left(\frac{z^{n+1}f(z)}{n!}\right)^{(n)}.$$

In this paper it is proved that $J_{n+1}(\alpha) \subset J_n(\alpha)$ ($n \in \mathbb{N}$ and $\alpha \in [0, 1)$). Since $J_0(\alpha)$ is the class of meromorphically convex functions of order α, $\alpha \in [0, 1)$, all functions in $J_n(\alpha)$ are meromorphically convex functions of order α. Further we consider the integrals of the functions in $J_n(\alpha)$.

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1 Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1}$$

which are regular in the punctured disc $U^* = \{z \,:\, 0 < |z| < 1\}$.

The Hadamard product of two functions $f,\ g \in \Sigma$ will be denoted by $f * g$. Let

$$D^n f(z) = \frac{1}{z(1-z)^{n+1}} * f(z) = \frac{1}{z}\left(\frac{z^{n+1} f(z)}{n!}\right)^{(n)} =$$

$$= \frac{1}{z} + (n+1)a_0 + \frac{(n+1)(n+2)}{2!}\, a_1 z + ...$$

In this paper among other things we shall show that a function f in Σ, which satisfies one of the conditions

$$\operatorname{Re}\left\{\frac{(D^{n+1} f(z))'}{(D^n f(z))'} - 2\right\} < -\frac{n+\alpha}{n+1}, \quad z \in U^*,\ \ n \in \mathbb{N}, \tag{2}$$

for $\alpha \in [0,1)$, is meromorphically convex of order α in U^*. More precisely, it is proved that for the classes $J_n(\alpha)$ of functions in Σ satisfying (2)

$$J_{n+1}(\alpha) \subset J_n(\alpha), \quad (n \in \mathbb{N},\ \ \alpha \in [0,1)) \tag{3}$$

holds. Since $J_0(\alpha) = \Sigma_k(\alpha)$ (the class of meromorphically convex functions of order α, $\alpha \in [0,1)$) the convexity of the members of $J_n(\alpha)$ is a consequence of (3).

We note that in [1] Aouf and Hossen obtained a new criterion for meromorphic univalent functions via the basic inclusion relationship $M_{n+1}(\alpha) \subset$

$M_n(\alpha)$, $n \in \mathbb{N}$ and $\alpha \in [0, 1)$, where $M_n(\alpha)$ is the class consisting of functions in Σ satisfying

$$\mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)} - 2\right\} < -\frac{n+\alpha}{n+1}, \quad z \in U^*,$$

$n \in \mathbb{N}$ and $\alpha \in [0, 1)$.

Futher, for $c \in (0, \infty)$ let

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\, dt.$$

In this paper it is shown that $F \in J_n(\alpha)$ whenever $f \in J_n(\alpha)$. Also it is shown that if $f \in J_n(\alpha)$, then

$$F(z) = \frac{n+1}{z^{n+2}} \int_0^z t^{n+1} f(t)\, dt. \tag{4}$$

belongs to $J_{n+1}(\alpha)$. Some known results of Bajpai [2], Goel and Sohi [3], Uralegaddi and Ganigi [10] and Sălăgean [8] are extended. Techniques are similar to those in [6] (see also [4], [5] and [8]).

2 The classes $J_n(\alpha)$

In order to prove our main results (Theorem 1 and Theorem 2 below) we shall need the following lemma due to S. S. Miller and P. T. Mocanu [4], [5] (see also [6] or [8])

Lemma A. *Let the function* $\Psi : \mathbb{C}^2 \to \mathbb{C}$ *satisfy*

$$\mathrm{Re}\, \Psi(ix, y) \leq 0$$

for all real x *and for all real* y, $y \leq -(1 + x^2)/2$. *If* $p(z) = 1 + p_1 z + ...$ *is analytic in the unit disc* $U = \{z : |z| < 1\}$ *and*

$$\mathrm{Re}\, \Phi(p(z), zp'(z)) > 0, \quad z \in U,$$

then $\operatorname{Re} p(z) > 0$ *for* $z \in U$.

Theorem 1. $J_{n+1}(\alpha) \subset J_n(\alpha)$ *for each integer* $n \in \mathbb{N}$ *and* $\alpha \in [0,1)$.

Proof. Let f be in $J_{n+1}(\alpha)$. Then

$$\operatorname{Re}\left\{\frac{(D^{n+2}f(z))'}{(D^{n+1}f(z))'} - 2\right\} < -\frac{n+1+\alpha}{n+2}, \quad z \in U^*. \tag{1}$$

We have to show that (1) implies the inequality

$$\operatorname{Re}\left\{\frac{(D^{n+1}f(z))'}{(D^{n}f(z))'} - 2\right\} < -\frac{n+\alpha}{n+1}, \quad z \in U^*. \tag{2}$$

We define the regular in $U = \{z : |z| < 1\}$ function p by

$$p(z) = \frac{n+2-\alpha}{1-\alpha} - \frac{n+1}{1-\alpha}\frac{(D^{n+1}f(z))'}{(D^{n}f(z))'}, \quad z \in U^*$$

and $p(0) = 1$. We have

$$\frac{(D^{n+1}f(z))'}{(D^{n}f(z))'} = \frac{1}{n+1}(n+2-\alpha-(1-\alpha)p(z)). \tag{3}$$

Differentiating (3) logarithmically, we obtain

$$\frac{(D^{n+1}f(z))''}{(D^{n+1}f(z))'} - \frac{(D^{n}f(z))''}{(D^{n}f(z))'} = \frac{-(1-\alpha)zp'(z)}{n+2-\alpha-(1-\alpha)p(z)}. \tag{4}$$

From the identity

$$z(D^n f(z))' = (n+1)D^{n+1}f(z) - (n+2)D^n f(z) \tag{5}$$

we have

$$z(D^n f(z))'' = (n+1)(D^{n+1}f(z))' - (n+3)(D^n f(z))'. \tag{6}$$

Using the identity (6) the equation (4) may be written

$$(n+2)\left\{\frac{(D^{n+2}f(z))'}{(D^{n+1}f(z))'} - 2\right\} + n+1+\alpha+(1-\alpha)p(z) = \frac{-(1-\alpha)zp'(z)}{n+2-\alpha-(1-\alpha)p(z)}$$

or

(7)
$$p(z)+\frac{zp'(z)}{n+2-\alpha-(1-\alpha)p(z)}=\frac{n+2}{1-\alpha}\left[2-\frac{(D^{n+2}f(z))'}{(D^{n+1}f(z))'}-\frac{n+1+\alpha}{n+2}\right]$$

Since $f\in J_{n+1}(\alpha)$, from (1) and (7) we have

$$\mathrm{Re}\left\{p(z)+\frac{zp'(z)}{n+2-\alpha-(1-\alpha)p(z)}\right\}>0,\quad z\in U. \tag{8}$$

We define the function Ψ by

$$\Psi(u,v)=u+v/(n+2-\alpha-(1-\alpha)u). \tag{9}$$

In order to use Lemma A we must verify that $\mathrm{Re}\,\Psi(ix,y)\leq 0$ whenever x and y are real numbers such that $y\leq-(1+x^2)/2$. We have

$$\mathrm{Re}\,\Psi(ix,y)=y\,\mathrm{Re}\,\frac{1}{n+2-\alpha-(1-\alpha)xi}=y\,\frac{n+2-\alpha}{(n+2-\alpha)^2+(1-\alpha)^2x^2}\leq$$
$$\leq-\frac{(1+x)^2\,(n+2-\alpha)}{(n+2-\alpha)^2+(1-\alpha)^2x^2}<0.$$

From (8) and (9) we obtain

$$\mathrm{Re}\,\Psi(p(z),zp'(z))>0,\quad z\in U.$$

By Lemma A we conclude that $\mathrm{Re}\,p(z)>0$ in U and (see (5)) this implies the inequality (2).

3 Integral operators

Theorem 2. *Let f be a function in Σ; if for a given $n\in\mathbb{N}$ and $c\in(0,\infty)$ f satisfies the condition*

$$\mathrm{Re}\left\{\frac{(D^{n+1}f(z))'}{(D^nf(z))'}-2\right\}<\frac{1-\alpha-2(n+\alpha)(c+1-\alpha)}{2(n+1)(c+1-\alpha)},\quad z\in U^*, \tag{10}$$

then

$$F(z)=\frac{c}{z^{c+1}}\int_0^z t^c f(t)\,dt \tag{11}$$

belongs to $J_n(\alpha)$.

Proof. Using the identity

$$z(D^nF(z))'=c(D^nf(z))-(c+1)(D^nF(z)) \tag{12}$$

obtained from (11) and the identity (5) for the function F the condition (10) may be written

(13)

$$\operatorname{Re}\left\{\frac{(n+2)\dfrac{(D^{n+2}F(z))'}{(D^{n+1}F(z))'}-n-2+c}{(n+1)-(n+1-c)\dfrac{(D^nF(z))'}{(D^{n+1}F(z))'}}\right\}-2<\frac{1-\alpha-2(n+\alpha)(c+1-\alpha)}{2(n+1)(c+1-\alpha)}.$$

We have to prove that (13) implies the inequality

$$\operatorname{Re}\left\{\frac{(D^{n+1}F(z))'}{(D^nF(z))'}-2\right\}<-\frac{n+\alpha}{n+1},\qquad z\in U^*. \tag{14}$$

We define the regular in U function p by

$$p(z)=\frac{n+2-\alpha}{1-\alpha}-\frac{n+1}{1-\alpha}\,\frac{(D^{n+1}F(z))'}{(D^nF(z))'},\quad z\in U^* \tag{15}$$

and $p(0)=1$.

From (6) we have

$$\frac{(D^{n+1}F(z))'}{(D^nF(z))'}=\frac{n+2-\alpha-(1-\alpha)p(z)}{n+1} \tag{16}$$

and for the function F instead of f the identity (15) becomes

$$z(D^nF(z))''=(n+1)(D^{n+1}F(z))'-(n+3)(D^nF(z))'. \tag{17}$$

Differentiating (16) logarithmically and using (17) we obtain

$$\frac{(n+2)\dfrac{(D^{n+2}F(z))'}{(D^{n+1}F(z))'}-n-2+c}{(n+1)-(n+1-c)\dfrac{(D^{n}F(z))'}{(D^{n+1}F(z))'}}-2=$$

$$=\frac{1}{n+1}\left[-n-\alpha-(1-\alpha)p(z)-\frac{(1-\alpha)zp'(z)}{1-\alpha+c-(1-\alpha)p(z)}\right]$$

and (13) is equivalent to

(18)
$$\frac{1-\alpha}{n+1}\,\mathrm{Re}\left\{p(z)+\frac{zp'(z)}{1-\alpha+c-(1-\alpha)p(z)}+\frac{1}{2(c+1-\alpha)}\right\}>0,\quad z\in U.$$

We define the function Ψ by

$$\Psi(u,v)=\frac{1-\alpha}{n+1}\left[u+\frac{v}{1-\alpha+c-(1-\alpha)u}+\frac{1}{2(c+1-\alpha)}\right]$$

For the real numbers $x,\ y$ with $y\le -(1+x^2)/2,$ we have

$$\mathrm{Re}\,\Psi(ix,y)=\frac{1-\alpha}{n+1}\left[\frac{1}{2(c+1-\alpha)}+\ \mathrm{Re}\,\frac{y}{c+1-\alpha-(1-\alpha)x\,i}\right]=$$

$$=\frac{1-\alpha}{n+1}\left[\frac{1}{2(c+1-\alpha)}-\frac{(1+x^2)(c+1-\alpha)}{2\left[(c+1-\alpha)^2+(1-\alpha)^2x^2\right]}\right]=$$

$$=\frac{1-\alpha}{n+1}\,\frac{-2(1-\alpha)c-c^2}{2\left[(c+1-\alpha)^2+(1-\alpha)^2x^2\right](c+1-\alpha)}<0.$$

We obtained that Ψ satisfies the conditions of Lemma A. This implies $\mathrm{Re}\,p(z)>0,$ and from (15) it follows that (12) implies (14), that is $F\in J_n(\alpha)$.

Putting $n=0$ in the statement of Theorem 2 we obtain the following result

Corollary 1. *If* $f\in\Sigma$ *and satisfies*

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}<\frac{1-\alpha-2\alpha(c+1-\alpha)}{2(c+1-\alpha)},$$

then F *given by (11) belongs to* $\Sigma_k(\alpha)$.

Putting $\alpha = 0$ and $c = 1$ in Corollary 1 we obtain the following result

Corollary 2. *If* $f \in \Sigma$ *and satisfies*

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{1}{4},$$

then the function F *defined by*

$$F(z) = \frac{1}{z^2}\int_0^z tf(t)\,dt$$

belongs to $\Sigma_k(0)$.

Remark 1. Corollary 1 was obtained by Goel and Sohi [3]. In [8] there is another extention of this result.

Remark 2. Corollary 2 extends a result of Bajpai [2]

Theorem 3. *If* $f \in J_n(\alpha)$, *then*

$$F(z) = \frac{n+1}{z^{n+2}}\int_0^z t^{n+1}f(t)\,dt$$

belongs to $J_{n+1}(\alpha)$.

Proof. From (11) we have

$$cD^n f(z) = (n+1)D^{n+1}F(z) - (n+1-c)D^nF(z)$$

and

$$cD^{n+1}f(z) = (n+2)D^{n+2}F(z) - (n+2-c)D^{n+1}F(z).$$

Taking $c = n+1$ in the above relations we obtain

$$\frac{(n+2)(D^{n+2}F(z))' - (D^{n+1}F(z))'}{(n+1)(D^{n+1}F(z))'} = \frac{(D^{n+1}f(z))'}{(D^nf(z))'}$$

which reduces to

$$\frac{(n+2)(D^{n+2}F(z))'}{(n+1)(D^{n+1}F(z))'} - \frac{1}{n+1} = \frac{(D^{n+1}f(z))'}{(D^nf(z))'}$$

Thus

$$\operatorname{Re}\left\{\frac{(n+2)(D^{n+2}F(z))'}{(n+1)(D^{n+1}F(z))'}-\frac{1}{n+1}-2\right\} = \operatorname{Re}\left\{\frac{(D^{n+1}f(z))'}{(D^{n}f(z))'}-2\right\} < -\frac{n+\alpha}{n+1}$$

from which it follows that

$$\operatorname{Re}\left\{\frac{(D^{n+2}F(z))'}{(D^{n+1}F(z))'}-2\right\} < -\frac{n+\alpha+1}{n+2}.$$

This completes the proof of Theorem 3.

Remark 3. Taking $\alpha = 0$ in the above theorems we get the results obtained by Uralegaddi and Ganigi [10].

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