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On the Rate of Convergence for Bivariate Beta Operators

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

In the present paper, we introduce the bivariate Beta operators and study the rate of convergence for the bivariate Beta operators.

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1 Introduction

A family of linear positive operators, from a mapping $C\,[0,\infty)$ into $C\,[0,\infty)$, the class of all bounded and continuous functions on $[0,\infty)$, is called Beta operators which is denoted by B_n and defined as

$$\left(B_n f\right)(x)=\frac{1}{n}\sum_{k=0}^{\infty}b_{n,k}(x)f\left(\frac{k}{n+1}\right),$$

where

$$b_{n,k}(x)=\frac{1}{\beta(k+1,n)}\frac{x^k}{(1+x)^{n+k+1}},\qquad x\in[0,\infty)$$

$\beta(k+1,n)$ denotes the Beta function given by $\Gamma(k+1).\Gamma(n)/\Gamma(k+n+1)$.

The Durrmeyer variant of these operators was studied by Gupta and Ahmad [2]. Very recently Deo [1] has studied some direct results for the operators B_n. We now introduce the bivariate Beta operators as

$$B_n(f,x,y) = \frac{1}{n}\sum_{k=0}^{\infty}\frac{1}{n+k}\sum_{m=0}^{\infty}\frac{1}{\beta(k+1,n)}\frac{1}{\beta(m+1,n+k)} \cdot\frac{x^k y^m}{(1+x+y)^{n+k+m+1}} f\left(\frac{k}{n+1},\frac{m}{n+1}\right) \tag{1.1}$$

where

$$(x,y)\in[0,\infty)\times[0,\infty)\equiv R_+^2 \text{ and } f\in C([0,\infty)\times[0,\infty))$$

In the present paper, we study the rate of convergence of these two dimensional Beta operators $B_n(f,x,y)$ by using the multivariate decomposition skills and the results of one dimensional Beta operators.

2 Basic Results

In this section we present some notational convention, definitions and basis results which are necessary to prove the main result.
For a function defined on the interval $[0,\infty)$ we set the following notations

$$C_{a,b,c,d}(R_+^2) = \{f : f\in C(R_+^2),\ wf\in L_\infty(R_+^2)\}$$
$$C^0_{a,b,c,d}(R_+^2) = \{f : f\in C_{a,b,c,d}(R_+^2), f(x,0)=f(0,y)=0\}$$

and

$$w(x,y) = x^a(1+x)^b\left(\frac{y}{1+x}\right)^c\left(1+\frac{y}{1+x}\right)^d$$

where $0<a,\ c<1;\ b,d<0;\ w(x)=x^a(1+x)^b$ and norm is defined as

$$\|wf\|_\infty = sup\,|wf|\,.$$

Also the weighted norm is given by

$$\|f\|_w = \|wf\|_\infty + |f(x,0)| + |f(0,y)|.$$

We define the Peetre's K-functional as

$$K_{t,\phi^\lambda}(f,t) = \inf_{g\in D}\{\|f-g\|_w + t\Phi(g)\},$$

where

$$D = \left\{g : g \in C(R_+^2), \Phi(g) < \infty, g_x, g_y \in A.C_{loc}\right\},$$

and with $\phi^2(x) = x(1+x)$

$$\Phi(g) = max\{\left\|\phi^{2\lambda} g_{xx}\right\|_\omega, \left\|\phi^{2\lambda} g_{yy}\right\|_\omega, \ \left\|\phi^{2\lambda} g_{xy}\right\|_\omega\}.$$

Through the present paper C denotes the positive constant not necessary the same at each occurrence.

Lemma 2.1. *For the bivariate operators $B_n(f,x,y)$, we have*

$$B_n(f,x,y) = \sum_{k=0}^{\infty} b_{n,k}(x) \sum_{m=0}^{\infty} b_{n+k,m}\left(\frac{y}{1+x}\right) f\left(\frac{k}{n+1}, \frac{m}{n+1}\right)$$
$$B_n(f,x,y) = \sum_{m=0}^{\infty} b_{n,m}(y) \sum_{k=0}^{\infty} b_{n+m,k}\left(\frac{x}{1+y}\right) f\left(\frac{k}{n+1}, \frac{m}{n+1}\right).$$

Proof. We have

$$\begin{aligned} B_n(f,x,y) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} b_{n,k,m}(x,y) f\left(\frac{k}{n+1}, \frac{m}{n+1}\right) \\ &= \sum_{k=0}^{\infty} b_{n,k}(x) \sum_{m=0}^{\infty} f\left(\frac{k}{n+1}, \frac{m}{n+1}\right) \frac{1}{(n+k)\beta(m+1,n+k)} \\ &\qquad \cdot \left(\frac{y}{1+x}\right)^m \left(1+\frac{y}{1+x}\right)^{-n-k-m-1} \\ &= \sum_{k=0}^{\infty} b_{n,k}(x) \sum_{m=0}^{\infty} f\left(\frac{k}{n+1}, \frac{m}{n+1}\right) b_{n+k,m}\left(\frac{y}{1+x}\right) \end{aligned}$$

The second assertion can be proved along the similar lines.

Remark 1. By using the properties of one dimensional Beta operators we have

$$\sum_{m=0}^{\infty} b_{n+k,m}\left(\frac{y}{1+x}\right) = 1 \text{ and } \sum_{k=0}^{\infty} b_{n+m,k}\left(\frac{x}{1+y}\right) = 1.$$

Lemma 2.2. *Suppose $n \in N$ and $(x,y) \in R_+^2$. Then it is easily verified from previous lemma that*

$$\begin{aligned}
B_n(1,x,y) &= 1 \\
B_n(s,x,y) &= x, && \text{for } f(s,t)=s, \\
B_n(t,x,y) &= y, && \text{for } f(s,t)=t, \\
B_n((s-x)^2,x,y) &= \frac{\phi^2(x)}{(n+1)}, \\
B_n((t-y)^2,x,y) &= \frac{\phi^2(y)}{(n+1)}, \\
\text{and} \quad B_n((s-x)(t-y),x,y) &= \frac{xy}{(n+1)}.
\end{aligned}$$

Remark 2. By Lemma 2.2, and using Holder's inequality we have

$$B_n(|s-x|\,,x,y) = O(\phi(x)n^{-1/2})$$

and

$$B_n(|t-y|\,,x,y) = O(\phi(y)n^{-1/2}).$$

Lemma 2.3. *(i) If $(x,y) \in [\frac{1}{n},\infty) \times [0,\infty)$, then*

$$B_n((s-x)^{2m},x,y) \le Cn^m(\phi(x))^{2m}$$

(ii) If $(x,y) \in [0,\infty) \times [\frac{1}{n},\infty)$, then

$$B_n((t-y)^{2m},x,y) \le Cn^m(\phi(y))^{2m}.$$

Proof. Clearly by Lemma 2.1, and using the property of one dimensional Beta operators, we have

$$\begin{aligned}
B_n((s-x)^{2m},x,y) &= \sum_{k=0}^{\infty}\left(\frac{k}{n+1}-x\right)^{2m} b_{n,k}(x)\sum_{m=0}^{\infty} b_{n+k,m}\left(\frac{y}{1+x}\right) \\
&= \sum_{k=0}^{\infty}\left(\frac{k}{n+1}-x\right)^{2m} b_{n,k}(x) \le Cn^m(\phi(x))^{2m}.
\end{aligned}$$

The proof of (ii) is similar.

Lemma 2.4. *For* $0 < \lambda < 1,\ \phi^2(u) = u(1+u),\ u \in [0,\infty)\,,\ t \in [0,\infty),$ *we have*

$$\left|\int_0^t |t-z|\,\phi^{-2\lambda}(z)dz\right| \le C(t-u)^2\left(\phi^{-2\lambda}(u) + u^{-\lambda}(1+u)^{-\lambda}\right) \tag{2.4}$$

Proof. Suppose $z = t + \mu(u-t),\ 0 \le \mu \le 1$, then

$$\begin{aligned}
\left|\int_0^t |t-z|\,\phi^{-2\lambda}(z)dz\right| &\le \int_0^t \frac{|t-z|}{z^\lambda}dz\left\{\frac{1}{(1+u)^\lambda} + \frac{1}{(1+t)^\lambda}\right\} \\
&\le \int_0^t \frac{\mu(t-u)^2}{(\mu u + (1-\mu)t)^\lambda}d\mu\left\{\frac{1}{(1+u)^\lambda} + \frac{1}{(1+t)^\lambda}\right\} \\
&\le (t-u)^2\int_0^t \frac{\mu^{1-u}}{u^\lambda}d\mu\left\{\frac{1}{(1+u)^\lambda} + \frac{1}{(1+t)^\lambda}\right\} \\
&\le \frac{1}{2-\lambda}(t-u)^2\left(\phi^{-2\lambda}(u) + u^{-\lambda}(1+u)^{-\lambda}\right).
\end{aligned}$$

The proof is completed.

Lemma 2.5.

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} b_{n,k,m}(x,y)\frac{\omega(x,y)}{\omega(\frac{k}{n+1},\frac{m}{n+1})} \le C. \tag{2.5}$$

Proof. From [3], we get

$$\sum_{k=1}^{\infty} b_{n,k}(x)\frac{\omega(x)}{\omega(\frac{k}{n+1})} \le C \ \text{ and } \ \sum_{k=1}^{\infty} b_{n,k}(x)\Big(\frac{\omega(x)}{\omega(\frac{k}{n+1})}\Big)^2 \le C$$

By using multivariate decompose skills, we obtain

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} b_{n,k,m}(x,y)\frac{\omega(x,y)}{\omega(\frac{k}{n+1},\frac{m}{n+1})} \\
&\qquad = \sum_{k=1}^{\infty} b_{n,k}(x)\frac{\omega(x)}{\omega(\frac{k}{n+1})}\sum_{k=1}^{\infty} b_{n+k,m}\left(\frac{y}{1+x}\right)\frac{\omega\left(\frac{y}{1+x}\right)}{\omega(\frac{m}{n+k+1})} \le C.
\end{aligned}$$

Lemma 2.6. *If* $f \in C^0_{a,b,c,d}(R^2_+)$, *then*

$$\big\|B_n(f)\big\|_\infty \le \big\|f\big\|_\infty. \tag{2.6}$$

Proof. Using Lemma 2.1 and Lemma 2.5, we have

$$\left|\omega(x,y)\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}b_{n,k,m}(x,y)f\left(\frac{k}{n+1},\frac{m}{n+1}\right)\right|$$

$$\leq \left\|\omega f\right\|_{\infty}\sum_{k=1}^{\infty}b_{n,k}(x)\frac{\omega(x)}{\omega\left(\frac{k}{n+1}\right)}\sum_{m=1}^{\infty}b_{n+k,m}\left(\frac{y}{1+x}\right)\frac{\omega\left(\frac{y}{1+x}\right)}{\omega\left(\frac{m}{n+k+1}\right)}\leq C\left\|\omega f\right\|_{\infty}.$$

The proof is completed.

3 Main Theorem

Theorem 3.1. *If* $f\in C^{0}_{a,b,c,d}(R^2_+),\ 0<\lambda<1$ *then*

$$\text{(3.1)}\quad \omega(x,y)\left|B_n(f,x,y)-f(x,y)\right|\leq C.K_{2,\phi^{\lambda}}(f,n^{-1}(\phi^{2(1-\lambda)}(x)+\phi^{2(1-\lambda)}(y)))$$

where C *is positive constant independent from* n,x,y.

Proof. Initially

$$B_n\left((s-x)^2(1+s)^{-\lambda};x,y\right)\leq Cn^{-1}\phi^2(x)(1+x)^{-\lambda}, \tag{3.2}$$

$$B_n\left((t-x)^2(1+t)^{-\lambda};x,y\right)\leq Cn^{-1}\phi(y)(1+y)^{-\lambda}, \tag{3.3}$$

$$B_n\left((s-x)(1+s)^{-\frac{\lambda}{2}}(t-y)(1-t)^{-\frac{\lambda}{2}},x,y\right) \leq C.n^{-1}\phi(x)\phi(y)(1+x)^{-\frac{\lambda}{2}}(1+y)^{-\frac{\lambda}{2}}. \tag{3.4}$$

By using Shwartz's inequality, Hölder inequality, Lemma 2.1 and Lemma 2.3, for $(x,y)\in I_n=[\frac{1}{n},\infty)\times[\frac{1}{n},\infty)$, we obtain

$$\begin{aligned}B_n\left((s-x)^2(1+s)^{-\lambda};x,y\right) &\leq \left(B_n(s-x)^4,x,y\right)^{\frac{1}{2}}\left(B_n(1+s)^{-2\lambda},x,y\right)^{\frac{1}{2}}\\ &\leq \left(B_n(s-x)^4,x,y\right)^{\frac{1}{2}}\left(B_n(1+s)^{-2},x,y\right)^{\frac{\lambda}{2}}\\ &\leq C.n^{-1}\phi^2(x)(1+x)^{-\lambda}.\end{aligned}$$

Now for $(x,y) \in I_n^c$, we get

$$\begin{aligned} B_n\big((s-x)^2(1+s)^{-\lambda}; x, y\big) &\le \big(B_n(s-x)^2, x, y\big) \\ &= n^{-1}\phi^2(x) \\ &= n^{-1}\phi^2(x)\frac{(1+x)^\lambda}{(1+x)^\lambda} \\ &\le 2^\lambda n^{-1}\phi^2(x)(1+x)^{-\lambda}. \end{aligned}$$

Similarly we can prove (3.3) and (3.4).

Now again using Lemma 2.1, Lemma 2.3 and (2.4), (3.1), (3.2), (3.3) and applying the Taylor's formula as well as Hardy-Littlewood majorant, for $g \in D$, we obtain

$$\begin{aligned} &\omega(x,y)\big|B_n(g,x,y) - g(x,y)\big| \\ &\quad\le w(x,y)\left|B_n\left(\int_x^s (s-\eta)\frac{\partial^2 g(\eta,y)}{\partial^2 x}d\eta, x, y\right)\right| \\ &\qquad + w(x,y)\left|B_n\left(\int_y^t (t-\xi)\frac{\partial^2 g(x,\xi)}{\partial^2 x}d\xi, x, y\right)\right| \\ &\qquad + w(x,y)\left|B_n\left(\int_x^s (s-\eta)\int_y^t (t-\xi)\frac{\partial^2 g(\eta,\xi)}{\partial^2 x}d\eta d\xi, x, y\right)\right| \\ &\quad\le C\Big[\big\|\omega\phi^{2\lambda}g_{xx}\big\|_\infty \phi^{-2\lambda}(x)B_n\big((s-x)^2, x, y\big) + x^{-\lambda}B_n\big((s-x)^2(1+s)^{-\lambda}, x, y\big) \\ &\qquad + \big\|\omega\phi^{2\lambda}g_{yy}\big\|_\infty \phi^{-2\lambda}(y)B_n\big((t-y)^2, x, y\big) + x^{-\lambda}B_n\big((t-y)^2(1+s)^{-\lambda}, x, y\big) \\ &\qquad + \big\|\omega\phi^{2\lambda}g_{xy}\big\|_\infty \phi^{-\lambda}(x)\phi^{-\lambda}(y)B_n\big((s-x)(t-y), x, y\big) \\ &\qquad + x^{-\lambda}B_n\big((t-x)^2(1+t)^{-\lambda}, x, y\big)\Big] \\ &\quad\le C.n^{-1}\big(\phi^{2(1-\lambda)}(x) + \phi^{2(1-\lambda)}(y) + \phi^{1-\lambda}(x)\phi^{1-\lambda}(y)\big)\Phi(g) \end{aligned}$$

$$\le C.n^{-1}\big(\phi^{2(1-\lambda)}(x) + \phi^{2(1-\lambda)}(y)\big)\Phi(g). \tag{3.5}$$

Thus, from (2.6) and (3.5), for $g \in D$ *and* $f \in C^0_{a,b,c,d}(R^2_+)$ we have

$$\begin{aligned} \omega(x,y)\big|B_n(f,x,y) - f(x,y)\big| &\le \big|\omega(x,y)B_n\big((f-g), x, y\big)\big| + \omega(x,y)\big|f(x,y) - g(x,y)\big| \\ &\quad + \omega(x,y)\big|B_n(g,x,y) - g(x,y)\big| \\ &\le C.K_{2,\phi^\lambda}\big(f, n^{-1}(\phi^{2(1-\lambda)}(x) + \phi^{2(1-\lambda)}(y))\big). \end{aligned}$$

The proof is completed.

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