General Mathematics Vol. 13, No. 3 (2005), 61–70

Generalized inverses of power means

Iulia Costin and **Gheorghe Toader**

Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

We look after the generalized inverses of power means in the family of Gini means and in the family of extended means.

2000 Mathematics Subject Classification: 26E60.

Keywords: Gini means; power means; generalized inverse of a mean.

1 Means

Usually the means are given by the following

Definition 1. *A* ***mean*** *is a function* $M : \mathbb{R}_+^2 \to \mathbb{R}_+$*, with the property*

$$\min(a, b) \le M(a, b) \le \max(a, b),\ \forall a, b > 0\ .$$

The mean M *is called* ***symmetric*** *if*

$$M(a, b) = M(b, a),\ \forall a, b > 0.$$

Each mean is **reflexive**, that is

$$M(a,a)=a,\ \forall a>0,$$

which will be used also as definition of $M(a,a)$ if it is necessary.

In what follows we use the **extended mean** (for $r\cdot s\cdot(r-s)\neq 0$)

$$E_{r,s}(a,b)=\left(\frac{s}{r}\cdot\frac{a^r-b^r}{a^s-b^s}\right)^{\frac{1}{r-s}}$$

and weighted Gini means defined by

$$\mathcal{B}_{r,s;\lambda}(a,b)=\left[\frac{\lambda\cdot a^r+(1-\lambda)\cdot b^r}{\lambda\cdot a^s+(1-\lambda)\cdot b^s}\right]^{\frac{1}{r-s}},\ r\neq s$$

with $\lambda\in[0,1]$ fixed. Weighted Lehmer means, $\mathcal{C}_{r;\lambda}=\mathcal{B}_{r,r-1;\lambda}$ and weighted power means $\mathcal{P}_{r,\lambda}=\mathcal{B}_{r,0;\lambda}$ are also used. We can remark that $\mathcal{P}_{0,\lambda}=\mathcal{G}_\lambda=\mathcal{B}_{r,-r;\lambda}$ is the weighted geometric mean. Also

$$\mathcal{B}_{r,s;0}=\mathcal{C}_{r;0}=\mathcal{P}_{r,0}=\Pi_2 \text{ and } \mathcal{B}_{r,s;1}=\mathcal{C}_{r;1}=\mathcal{P}_{r,1}=\Pi_1\ ,$$

where we denote by Π_1 and Π_2 the first respectively the second projection defined by

$$\Pi_1(a,b)=a,\ \Pi_2(a,b)=b,\ \forall a,b\geq 0.$$

Given three means M,N and P , the expression

$$P(M,N)(a,b)=P(M(a,b),N(a,b)),\ \forall a,b>0,$$

defines also a mean $P(M,N)$. Using it we can give the following

Definition 2. *The mean N is called $P-$ complementary to M if*

$$P(M,N)=P.$$

If the $P-$ complementary of M exists and is unique, we denote it by M^P.

Proposition 3. *For every mean M we have*

$$M^M = M \ , \ \Pi_1^M = \Pi_2 \ , \ M^{\Pi_2} = \Pi_2$$

and if P is a symmetric mean then

$$\Pi_2^P = \Pi_1 \ .$$

Remark 4. *In what follows, we shall call these results as* ***trivial cases*** *of complementariness.*

More comments on this notion and its importance in the determination of the limit of a double sequence can be found in [5] or [6]. We study the complementariness with respect to the weighted geometric mean $\mathcal{G}_\lambda = \mathcal{P}_{0,\lambda}$. We denote the $\mathcal{G}_\lambda-$ complementary of M by $M^{\mathcal{G}(\lambda)}$ and we call it generalized inverse of M. We omit to write λ if it is equal with $1/2$. Of course

$$M^{\mathcal{G}(\lambda)} = \left(\frac{\mathcal{G}_\lambda}{M^\lambda}\right)^{\frac{1}{1-\lambda}} ,$$

but it is not always a mean. For instance, taking $M = \mathcal{G}_\mu$, obviously there exists a ν such that $M^{\mathcal{G}(\lambda)} = \mathcal{G}_\nu$. More exactly

$$\mathcal{G}_\mu^{\mathcal{G}(\lambda)} = \mathcal{G}_{\frac{\lambda(1-\mu)}{1-\lambda}} ,$$

but it is a mean if and only if

$$\mu \geq 2 - \frac{1}{\lambda}, \ \lambda \in \left[\frac{1}{2}, 1\right] .$$

For other means it is more difficult to determine the complementary. In what follows we present a method which can be useful in some cases.

2 Series expansion of means

For the study of some problems related to means in [4] is used their power series expansion. In fact, for a mean M is considered the series of the normalized functions $M(1,1-x),\ x\in(0,1)$.

For example, in [3] is proved that the extended mean $E_{r,s}$ has the following first terms of the power series expansion

$$E_{r,s}(1,1-x)=1-\frac{1}{2}\cdot x+\frac{r+s-3}{24}\cdot x^2+\frac{r+s-3}{48}\cdot x^3$$

$$-\left[2(r^3+r^2s+rs^2+s^3)-5(r+s)^2-70(r+s)+225\right]\cdot\frac{x^4}{5760}$$

$$-\left[2(r^3+r^2s+rs^2+s^3)-5(r+s)^2-30(r+s)+105\right]\cdot\frac{x^5}{3840}+\cdots.$$

Also in [2] is given the series expansion of the weighted Gini mean

$$\mathcal{B}_{q,q-r;\nu}(1,1-x)=1-(1-\nu)\cdot x+\nu\,(1-\nu)\,(2q-r-1)\cdot\frac{x^2}{2!}-\nu\,(1-\nu)$$

$$\cdot\{\nu[6q^2-6q\,(r+1)+(r+1)\,(2r+1)]-3q\,(q-r)-(r-1)\,(r+1)\}\cdot\frac{x^3}{3!}$$

$$-\nu\,(1-\nu)\cdot\{\nu^2[-24q^3+36q^2\,(r+1)-12q\,(r+1)\,(2r+1)+(r+1)\,(2r+1)$$

$$\cdot\,(3r+1)]+\nu[24q^3-12q^2\,(3r+1)+12q\,(r+1)\,(2r-1)-3\,(r+1)\,(2r+1)$$

$$\cdot\,(r-1)]-4q^3+6q^2\,(r-1)-2q\left(2r^2-3r-1\right)+(r-2)\,(r-1)\,(r+1)\}$$

$$\cdot\frac{x^4}{4!}-\nu\,(1-\nu)\cdot\left\{\nu^3\left[120q^4-240q^3\,(r+1)+120q^2(r+1)(2r+1)\right.\right.$$

$$-20q(r+1)(2r+1)(3r+1)+(r+1)(2r+1)(3r+1)(4r+1)]$$

$$+\nu^2\left[-180q^4+180q^3(2r+1)-90q^2(r+1)(4r-1)+30q(r+1)(2r+1)\right.$$

$$\cdot(3r-2)-6(r-1)(r+1)(2r+1)(3r+1)]+\nu\left[70q^4-20q^3(7r-2)+10q^2\right.$$

$$\cdot(14r^2-6r-9)-10q(r+1)(7r^2-12r+3)+(r-1)(2r+1)(7r-11)(r+1)]$$

$$-5q^4 + 10q^3(r-2) - 5q^2(2r^2 - 6r + 3) + 5q(r-2)(r^2 - 2r - 1)$$
$$-(r+1)(r-1)(r-2)(r-3)\} \cdot \frac{x^5}{5!} + \cdots .$$

In the special case $q = r = p$ we get the series expansion of the weighted power means

$$\mathcal{P}_{p;\mu}(1, 1-x) = 1 - (1-\mu) \cdot x + \mu(1-\mu)(p-1) \cdot \frac{x^2}{2} + \mu(1-\mu)(p-1)$$
$$\cdot \left[p\,(1-2\mu) + \mu + 1\right] \cdot \frac{x^3}{6} + \mu(1-\mu)(p-1)[p^2 \left(6\mu^2 - 6\mu + 1\right)$$
$$-p\left(5\mu^2 + 3\mu - 3\right) + \mu^2 + 3\mu + 2] \cdot \frac{x^4}{24} + \mu(1-\mu)(p-1)$$
$$\cdot [p^3 \left(24\mu^3 - 36\mu^2 + 14\mu - 1\right) - p^2 \left(26\mu^3 + 6\mu^2 - 29\mu + 6\right)$$
$$p\left(9\mu^3 + 24\mu^2 + 4\mu - 11\right) - \mu^3 - 6\mu^2 - 11\mu - 6] \cdot \frac{x^5}{120} + \cdots .$$

3 Generalized inverses of power means

In [1] was proved the following

Theorem 5. *The first terms of the series expansion of the generalized inverse of $P_{p,\mu}$ are*

$$\mathcal{P}_{p,\mu}^{\mathcal{G}(\lambda)}(1, 1-x) = 1 - \left[1 - \alpha\,(1-\mu)\right] x - \frac{\alpha}{2!}\,(1-\mu)\left[1 + \mu p - \alpha\,(1-\mu)\right] x^2$$
$$+\frac{\alpha}{3!}\,(1-\mu)\left[\alpha^2\,(1-\mu)^2 - 3\mu p\alpha\,(1-\mu) + \mu p^2\,(2\mu-1) - 1\right] x^3$$
$$-\frac{\alpha}{4!}\,(1-\mu)\left\{-\alpha^3\,(1-\mu)^3 + 2\alpha^2\,(1-\mu)^2\,(3\mu p - 1) - \alpha\,(1-\mu)\right.$$
$$\left.\cdot \left[p\mu\,(11p\mu - 4p - 6) - 1\right] + p^3\mu\left(6\mu^2 - 6\mu + 1\right) - 2p^2\mu\,(2\mu - 1) - p\mu + 2\right\}$$
$$\cdot x^4 + \frac{\alpha}{5!}\,(1-\mu)\left\{\alpha^4\,(1-\mu)^4 - 5\alpha^3\,(1-\mu)^3\,(2p\mu - 1) + 5\alpha^2\,(1-\mu)^2\right.$$

$$\cdot [p\mu (7p\mu - 2p - 6) + 1] - 5\alpha (1 - \mu) \left[p^3\mu \left(10\mu^2 - 8\mu + 1\right)\right.$$

$$-p\mu (11p\mu - 4p - 3) + 1] + p^4\mu \left(24\mu^3 - 36\mu^2 + 14\mu - 1\right) - 5p^3\mu$$

$$\left.\cdot \left(6\mu^2 - 6\mu + 1\right) + 5p\mu (2p\mu - p + 1) - 6\right\} \cdot x^5 + \cdots ,$$

where $\alpha = \frac{\lambda}{1-\lambda}$.

Using it, we can prove the following result.

Theorem 6. *The relation*

$$\mathcal{P}_{p,\mu}^{\mathcal{G}(\lambda)} = \mathcal{B}_{q,q-r;\nu}$$

holds if and only if we are in one of the following cases:

(i) $\mathcal{P}_{p,\mu}^{\mathcal{G}(0)} = \mathcal{B}_{q,q-r;0}$;
(ii) $\mathcal{P}_{p,1}^{\mathcal{G}(\lambda)} = \mathcal{B}_{q,q-r;0}$;
(iii) $\mathcal{P}_{p,0}^{\mathcal{G}} = \mathcal{B}_{q,q-r;1}$;
(iv) $\mathcal{P}_{p,0}^{\mathcal{G}(0)} = \mathcal{B}_{q,q-r;0}$;
(v) $\mathcal{P}_{p,0}^{\mathcal{G}(1/3)} = \mathcal{B}_{q,-q;1/2}$;
(vi) $\mathcal{P}_{0,(3\lambda-1)/2\lambda}^{\mathcal{G}(\lambda)} = \mathcal{B}_{q,-q;1/2},\ \lambda \geq 1/3;$
(vii) $\mathcal{P}_{0,(2\lambda-1)/\lambda}^{\mathcal{G}(\lambda)} = \mathcal{B}_{q,-q;1},\ \lambda \geq 1/2;$
(viii) $\mathcal{P}_{p,\mu}^{\mathcal{G}} = \mathcal{B}_{-p,0;1-\mu}$ *;*
(ix) $\mathcal{P}_{p,\mu}^{\mathcal{G}} = \mathcal{B}_{0,-p;1-\mu}$.

Proof. Equating the coefficients of x , in $\mathcal{P}_{p,\mu}^{\mathcal{G}(\lambda)}(1, 1-x)$ and in $\mathcal{B}_{q,q-r;\nu}(1, 1-x)$ we have the condition

$$\nu = \alpha (1 - \mu) . \tag{1}$$

Then, the equality of the coefficients of x^2 gives the condition

$$\alpha (1 - \mu) [\mu p + (1 - \alpha + \alpha\mu) (2q - r)] = 0. \tag{2}$$

Let us consider the following cases: a) $\alpha = 0$, which implies $\nu = 0$ and so the relation (i); b) $\mu = 1$, which also implies $\nu = 0$ and so the relation (ii); c) $\mu = 0$, for which (1) and (2) implies $\nu = \alpha$ and $r = 2q$; passing to the equality of the coefficients of x^3, these relations imply

$$q^2 \nu (1 - \nu) (1 - 2\nu) = 0.$$

So we have to consider the special cases: c') $\nu = \alpha = 1$ which leads to (iii); c") $\nu = \alpha = 0$ which implies (iv) ; c"') $\nu = \alpha = 1/2$ which gives (v). Remark that $r = 2q \neq 0$, thus we pass to the case: d) $p = 0$ for which (2) implies $r = 2q$ and taking into account (1) , the coefficients of x^3 imply

$$q^2 \nu (1 - \nu) (2\nu - 1) = 0,$$

giving (vi) and (vii); e) Replacing $\nu = \alpha (1 - \mu)$ and $\mu p = (r - 2q) (1 - \alpha + \alpha\mu)$ in the coefficients of x^3, x^4 and x^5 , we get the special cases: e') $\alpha = 1, r = -p = q$, giving $(viii)$, and e") $\alpha = 1, r = p,\ q = 0$, that is (ix).

Corollary 7. *The relation*

$$\mathcal{P}_{p,\mu}^{\mathcal{G}(\lambda)} = \mathcal{B}_{q,q-r;\nu}$$

holds only in the following nontrivial cases:

(i) $\mathcal{P}_{p,0}^{\mathcal{G}(1/3)} = \mathcal{B}_{q,-q;1/2}$;
(ii) $\mathcal{P}_{0,(3\lambda-1)/2\lambda}^{\mathcal{G}(\lambda)} = \mathcal{B}_{q,-q;1/2},\ \lambda \geq 1/3;$
(iii) $\mathcal{P}_{0,(2\lambda-1)/\lambda}^{\mathcal{G}(\lambda)} = \mathcal{B}_{q,-q;1},\ \lambda \geq 1/2;$
(iv) $\mathcal{P}_{p,\mu}^{\mathcal{G}} = \mathcal{B}_{-p,0;1-\mu}$;
(v) $\mathcal{P}_{p,\mu}^{\mathcal{G}} = \mathcal{B}_{0,-p;1-\mu}$.

Corollary 8. *The relation*

$$\mathcal{P}_{p,\mu}^{\mathcal{G}} = \mathcal{B}_{q,q-r;\nu}$$

holds only in the following nontrivial cases:

$$(i)\ \mathcal{P}^{\mathcal{G}}_{p,\mu} = \mathcal{B}_{-p,0;1-\mu}\ ;$$
$$(ii)\ \mathcal{P}^{\mathcal{G}}_{p,\mu} = \mathcal{B}_{0,-p;1-\mu}\ .$$

Corollary 9. *The relation*

$$\mathcal{P}^{\mathcal{G}(\lambda)}_{p,\mu} = \mathcal{C}_{q;\nu}$$

holds only in the following nontrivial cases:

$$(i)\ \mathcal{P}^{\mathcal{G}(1/3)}_{p,0} = \mathcal{C}_{1/2;1/2}\ ;$$
$$(ii)\ \mathcal{P}^{\mathcal{G}(\lambda)}_{0,(3\lambda-1)/2\lambda} = \mathcal{C}_{1/2;1/2},\ \lambda \geq 1/3;$$
$$(iii)\ \mathcal{P}^{\mathcal{G}(\lambda)}_{0,(2\lambda-1)/\lambda} = \mathcal{C}_{q;1},\ \lambda \geq 1/2;$$
$$(iv)\ \mathcal{P}^{\mathcal{G}}_{-1,\mu} = \mathcal{C}_{1;1-\mu}\ ;$$
$$(v)\ \mathcal{P}^{\mathcal{G}}_{1,\mu} = \mathcal{C}_{0;1-\mu}\ .$$

Corollary 10. *The relation*

$$\mathcal{P}^{\mathcal{G}(\lambda)}_{p,\mu} = \mathcal{P}_{q,\nu}$$

holds only in the following nontrivial cases:

$$(i)\ \mathcal{P}^{\mathcal{G}(1/3)}_{p,0} = \mathcal{P}_{0,1/2}\ ;$$
$$(ii)\ \mathcal{P}^{\mathcal{G}(\lambda)}_{0,(3\lambda-1)/2\lambda} = \mathcal{P}_{0,1/2},\ \lambda \geq 1/3;$$
$$(iii)\ \mathcal{P}^{\mathcal{G}(\lambda)}_{0,(2\lambda-1)/\lambda} = \mathcal{P}_{q,1},\ \lambda \geq 1/2;$$
$$(iv)\ \mathcal{P}^{\mathcal{G}}_{p,\mu} = \mathcal{P}_{-p,1-\mu}\ .$$

Theorem 11. *The relation*

$$\mathcal{P}^{\mathcal{G}(\lambda)}_{p,\mu} = \mathcal{E}_{r,s}$$

holds if and only if we are in one of the following cases:

$$(i)\ \mathcal{P}^{\mathcal{G}(1/3)}_{p,0} = \mathcal{E}_{r,-r}\ ;$$
$$(ii)\ \mathcal{P}^{\mathcal{G}(\lambda)}_{0,\frac{3\lambda-1}{2\lambda}} = \mathcal{E}_{r,-r}\ ,\ \lambda \in \left[\tfrac{1}{3}, 1\right)\ ;$$
$$(iii)\ \mathcal{P}^{\mathcal{G}}_{p} = \mathcal{E}_{-p,-2p}\ .$$

Proof. Equating the coefficients of x , in $\mathcal{P}_{p,\mu}^{\mathcal{G}(\lambda)}(1,1-x)$ and in $\mathcal{E}_{r,s}(1,1-x)$, we have the condition

$$\alpha\,(1-\mu)=\frac{1}{2}. \tag{3}$$

The coefficients of x^2 give the condition

$$r+s=-6\mu p\ ,$$

and the coefficients of x^3 are equal if, moreover,

$$\mu\,(2\mu-1)\,p^2=0.$$

We consider the cases: a) $\mu=0$ which gives $\lambda=1/3$ and $s=-r$, thus (i); b) $p=0$ which implies $s=-r$ and from (3) we get

$$\mu=\frac{3\lambda-1}{2\lambda},\ \text{for}\ \frac{1}{3}\leq\lambda<1,$$

thus (ii); c) $\mu=1/2$ which gives $\lambda=1/2$ and $s=-r-3p$. Equating also the coefficients of x^4 , we obtain in this case:

$$p\,(p+r)\,(2p+r)=0.$$

We have the special cases: c') $p=0$, giving (ii); c") $r=-p$, thus $s=-2p$, so the case (iii); c"') $r=-2p$, so $s=-p$, thus again (iii) (because $\mathcal{E}_{r,s}=\mathcal{E}_{s,r}$). By direct computation, we verify that the three cases are valid.

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Department of Computer Science
Technical University of Cluj, Romania
E-mail: *Gheorghe.Toader@math.utcluj.ro*

Department of Mathematics
Technical University of Cluj, Romania