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A Class of Holomorphic Functions Defined Using a Differential Operator

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

By using the differential operator $D^n f(z)$, $z \in U$ (Definition 1), we introduce a class of holomorphic functions. We let $M_n(\alpha)$ denote this class and we obtain some inclusion relations.

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1 Introduction and preliminaries

Denote by U the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

Let $\mathcal{H}[U]$ be the space of holomorphic function in U.

We let:

$$A_n = \{f \in \mathcal{H}[U], \ f(z) = z + a_{n+1} z^{n+1} + \dots, \ z \in U\}$$

with $A_1 = A$.

Let

$$K = \left\{ f \in A : \ \mathrm{Re}\, \frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U \right\}$$

be the class of normalized convex functions in U.

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

We use the following subordination results.

Lemma A. (Hallenbeck and Ruscheweyh [1, p. 71]) *Let h be a convex function with $h(0) \equiv a$ and let $\gamma \in \mathbb{C}^*$ be a complex number with* Re $\gamma \geq 0$. *If $p \in \mathcal{H}[U]$ with $p(0) = a$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z)$$

then

$$p(z) \prec q(z) \prec h(z)$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt.$$

The function q is convex and is the best dominant. (The definition of best dominant is given in [2]).

Lemma B. (S. S. Miller and P. T. Mocanu [2]) *Let g be a convex function in U and let*

$$h(z) = g(z) + n\alpha z g'(z)$$

where $\alpha > 0$ and n is a positive integer.

If $p(z) = g(0) + p_n z^n + \dots$ is holomorphic in U and

$$p(z) + \alpha z p'(z) \prec h(z)$$

then

$$p(z) \prec g(z)$$

and this result is sharp.

Definition 1. (G. S. Sălăgean [4]). For $f \in A$ and $n \geq 0$ we define the operator $D^n f$ by

$$D^0 f(z) = f(z)$$

$$D^{n+1} f(z) = z[D^n f(z)]', \ z \in U.$$

2 Main results

Definition 2. For $\alpha < 1$ and $n \in \mathbb{N}$, we let $M_n(\alpha)$ denote the class of functions $f \in A$ which satisfy the inequality:

$$\text{Re } [D^n f(z)]' > \alpha.$$

Theorem 1. *If* $\alpha < 1$ *and* $n \in \mathbb{N}$, *then we have*

$$M_{n+1}(\alpha) \subset M_n(\delta),$$

where

$$\delta = \delta(\alpha) = 2\alpha - 1 + 2(1 - \alpha) \ln 2.$$

The result is sharp.

Proof. Let $f \in M_{n+1}(\alpha)$. We have

(1) $$D^{n+1} f(z) = z[D^n f(z)]', \quad z \in U.$$

Differentiating (1) we obtain

(2) $$[D^{n+1} f(z)]' = [D^n f(z)]' + z[D^n f(z)]''.$$

If we let $p(z) = [D^n f(z)]'$, then (2) becomes

$$[D^{n+1} f(z)]' = p(z) + zp'(z).$$

Since $f \in M_{n+1}(\alpha)$, by using Definition 2 we have

$$\text{Re}\,[p(z) + zp'(z)] > \alpha$$

which is equivalent to

$$p(z) + zp'(z) \prec \frac{1+(2\alpha-1)z}{1+z} \equiv h(z).$$

By using Lemma A, we have:

$$p(z) \prec q(z) \prec h(z)$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt = \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt$$

$$= \frac{1}{z}\int_0^z \left[2\alpha - 1 + 2(1-\alpha)\frac{1}{1+t}\right] dt = 2\alpha - 1 + 2(1-\alpha)\frac{\ln(1+z)}{z}.$$

Function q is convex and it is the best dominant.

From $p(z) \prec q(z)$, it results that

$$\text{Re}\, p(z) > \text{Re}\, q(1) = \delta = \delta(\alpha) = 2\alpha - 1 + 2(1-\alpha)\ln 2$$

from which we deduce $M_{n+1}(\alpha) \subset M_n(\delta)$.

Theorem 2. *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z).$$

If $f \in M_n(\alpha)$ and verifies the differential subordination

$$[D^{n+1}f(z)]' \prec h(z) \tag{3}$$

then

$$[D^n f(z)]' \prec g(z)$$

and this result is sharp.

Proof. From $D^{n+1}f(z) = z[D^n f(z)]'$, we obtain

$$[D^{n+1}f(z)]' = [D^n f(z)]' + z[D^n f(z)]''.$$

If we let $p(z) = [D^n f(z)]'$, $p'(z) = [D^n f(z)]''$ then we obtain

$$[D^{m+1}f(z)]' = p(z) + zp'(z)$$

and (3) becomes

$$p(z) + zp'(z) \prec g(z) + zg'(z) \equiv h(z).$$

By using Lemma B, we have

$$p(z) \prec g(z), \text{ i.e., } [D^m f(z)]' \prec g(z)$$

and this result is sharp.

Theorem 3. *Let g be a convex function, $g(0) = 1$, and*

$$h(z) = g(z) + zg'(z).$$

If $f \in M_n(\alpha)$ and verifies the differential subordination

$$[D^n f(z)]' \prec h(z), \quad z \in U, \tag{4}$$

then

$$\frac{D^n f(z)}{z} \prec g(z), \quad z \in U$$

and this result is sharp.

Proof. We let $p(z) = \dfrac{D^n f(z)}{z}$, $z \in U$, and we obtain

$$D^n f(z) = zp(z).$$

By differentiating, we obtain

$$[D^n f(z)]' = p(z) + zp'(z), \quad z \in U.$$

Then (4) becomes

$$p(z) + zp'(z) \prec h(z) \equiv g(z) + zg(z).$$

By using Lemma B, we have

$$p(z) \prec g(z),$$

i.e.

$$\frac{D^n f(z)}{z} \prec g(z).$$

We remark that in the case of meromorphic functions a similar results was obtained by M. Pap in [3].

References

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