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The regenerative process with rare event appears on the second phase [1]

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Abstract

In many application, the regeneration period is a sum of two random variables ξ and η, where ξ is a random period corresponding to the operation of the system in the absence of component failures (is called free period), and η is a random interval which starts with a failure of some component and which terminates by the return of the whole system to the brand new state; the end of η is a regeneration point. The η interval is called the busy period. System failures take place only in busy periods. In situations typical for reliability theory, the busy period is very small in comparison with the average length of the free period. This fact reflects the fast repair property. In this paper a two phase regenerative process will be analyzed and a limit theorem on the convergence of system time to failure to an exponential random variable will be presented.

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1 Two phase regenerative process

Let us consider a nonnegative random process $k(t)$. The time intervals on which $k(t) = 0$ will be the free periods and the time intervals on which $k(t) > 0$ will be the busy periods. Time instants at which $k(t)$ enters 0 form a regenerative process. In many applications, $k(t)$ is defined as the number of failed components in the system.
Let ξ_1 , ξ_2 , ..., and η_1^0 , η_2^0 , ... be the independent copies of free and busy periods, respectively. It will be assumed that $\xi_i \sim \text{Exp}(\lambda)$ (the random variable ξ_i has *exponential distribution* with parameter λ; i.e. $P(\xi_i < t) = 1 - e^{-\lambda t}$), which is a quite realistic assumption for a system whose components have exponentially distributed lifetimes. The time instants $t_i = t_{i-1} + \xi_i + \eta_i^0$, with $t_0 = 0$ $i = 1, 2, ...$ are regeneration points (see fig.1).

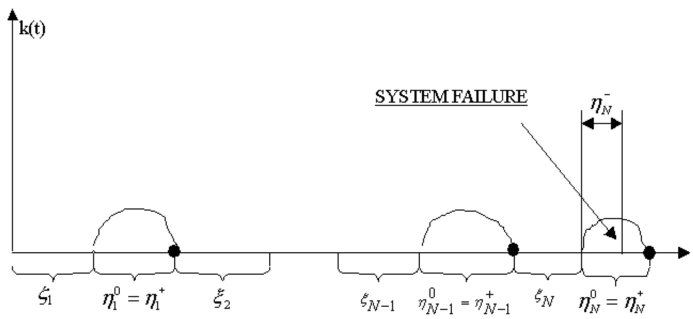

Figure 1: Two phase regenerative process; ξ_i and η_i^0 are the free and busy periods, respectively; • designates regeneration points

On each busy period, some event A designating system failure can occur. This event appears with probability p, independently of the history of $k(t)$

during previous regeneration cycles. When $k(t)$ counts the number of failed components in the system; A is defined usually as the crossing by $k(t)$ of some prescribed critical level.

Denote by η_1^+ , η_2^+ , ... the random lengths of those busy periods on which A did not appear. η_n^- denotes the time interval between the beginning of the busy period and the appearance of A. Now system lifetime τ has the following representation:

$$\tau = \xi_1 + \xi_2 + ... + \xi_N + \eta_1^+ + \eta_2^+ + ... + \eta_{N-1}^+ + \eta_N^-$$

where $N \sim \mathcal{G}(p)$ (the random variable N has *geometric distribution*), that is:

$$P(N = k) = (1-p)^{k-1}p \ , \quad k \geq 1$$

Set $\xi = \sum_{i=1}^{N} \xi_i$, $\tilde{\eta} = \sum_{i=1}^{N-1} \eta_i^+ + \eta_N^-$ and

$$\eta_n = \eta_n^+(1-\chi_n) + \eta_n^-\chi_n$$

where χ_n is the indicator of the event A:

$$\chi_n = \begin{cases} 1 & \text{if } A \text{ takes place on } \eta_n^0 \\ 0 & \text{otherwise} \end{cases}$$

We see that $\tau = \xi + \tilde{\eta}$, where $\xi \sim \mathrm{Exp}(\lambda p)$. It shows this fact.

How $\xi_i \sim \mathrm{Exp}(\lambda)$, then the Laplace transformation is:

$$\varphi(z) = E[e^{-z\xi_i}] = \int\limits_0^\infty e^{-zt}\mathrm{d}(1-e^{-\lambda t}) = \lambda \int\limits_0^\infty e^{-(\lambda+z)t}\mathrm{d}t = \frac{\lambda}{\lambda+z}$$

Also,

$$\begin{aligned} \mathcal{L}(\xi) &= E[e^{-z\xi}] = \sum_{k=1}^{\infty} P(N=k)\cdot E[e^{-z(\xi_1+\xi_2+...+\xi_N)} \mid N=k] \\ &= \sum_{k=1}^{\infty}(1-p)^{k-1}p\cdot[\varphi(z)]^k = \frac{p\,\varphi(z)}{1-(1-p)\varphi(z)} \qquad (1) \end{aligned}$$

Substituting it into (1), we obtain:

$$\mathcal{L}(\xi) = \left(1 + \frac{1}{p\lambda} z\right)^{-1}$$

which means that $\xi \sim$ Exp(λp) . Thus, one could expect that if the influence of the second summand $\tilde{\eta}$ is negligible, then the distribution of τ will be closely approximated by an exponential distribution. Fast repair means that $E[\eta_n^0] \ll E[\xi_n]$. This will be crucial to prove convergence to the exponential distribution. The following theorem establishes this convergence and also presents very useful bounds on the deviation of $P(\tau > t)$ from $e^{-\lambda pt}$. Let $E[\eta_n] = \mu^*$, $E[\eta_n^0] = \mu_0$ and $\bar{F}_t(x)$ be the conditional probability that $\tilde{\eta}$ exceeds x given that $\xi = t$.(In general, the busy period may not be independent on the preceding free period.)

Theorem 1. [1] Under the above notations:

$$e^{-\lambda pt} \leq P(\tau > t) \leq e^{-\lambda pt} + \lambda\mu^* \leq e^{-\lambda pt} + \lambda\mu_0 \tag{2}$$

Proof.

$$P(\tau > t) = P(\xi + \tilde{\eta} > t) \geq P(\xi > t) = e^{-\lambda pt}$$

The event $(\tau > t)$ happens either if $\xi > t$ or $\xi = x$, $x \in (0, t)$ and $\tilde{\eta}$ exceeds $t - x$ (see fig.1). Therefore,

$$P(\tau > t) = e^{-\lambda pt} + \int_0^t \bar{F}_x(t-x) \mathrm{d}(1 - e^{-\lambda px}) = e^{-\lambda pt} + \int_0^t \lambda p\, e^{-\lambda px} \bar{F}_x(t-x) \mathrm{d}x$$

To proceed with the proof, we need the following lemma:

Lemma 1. [1]

$$\int_0^t \lambda p\, e^{-\lambda px} \bar{F}_x(t-x) \mathrm{d}x \leq \lambda\mu^*$$

The proof is technical and we omit it. This proof there is in [1] and the proof of the theorem 1 is obtained by author.
Now note that on every trajectory of $k(t)$, $\eta_n \leq \eta_n^0$ and thus $\mu^* \leq \mu_0$. This proves the inequality of the theorem.

2 EXAMPLE - A system with reparable standby: arbitrary distribution of repair time

A system has one operating unit and n units in standby. The standby can be of various types (warm or cold). The lifetime of the operating unit and the units on standby (excluding the cold standby) are exponentially distributed. A unit which fails goes to the repair shop having r identical repair facilities (channels). Each channel restores the failed unit during time $\eta \sim G(x)$ (the random variable η has cumulative distribution function $G(x)$). After the repair is completed, the unit returns to the standby. If the operating unit fails, its place is taken by another one from the standby. In this way, all units circulate through operation, repair (with possible stay in the waiting line) and back to standby (see fig.2).

Let us show how various types of standby can be incorporated in the above model. It is easy to verify that:

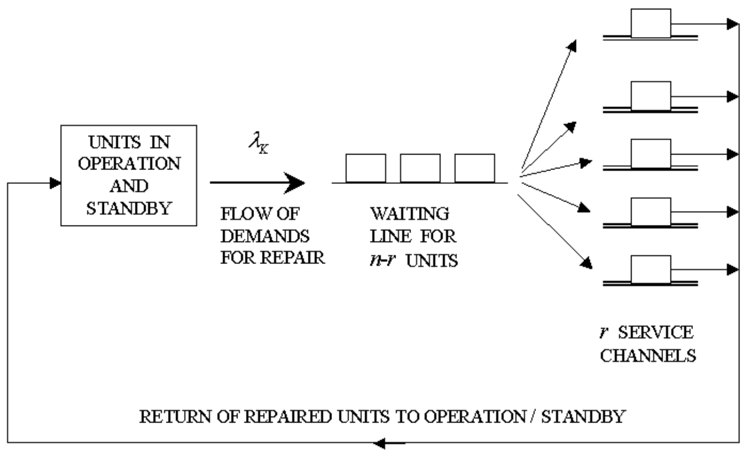

Figure 2: Closed service system: scheme of operation

$$\lambda_k = \begin{cases} \lambda & \text{if there is one operating unit and all other units are on cold standby, } \lambda \text{ is the failure rate of the operating unit} \\ \lambda(n+1-k) & \text{if all nonfailed units, including the operating unit, have failure rate } \lambda \\ \lambda + (n-k)\lambda^* & \text{if the operating unit has failure rate } \lambda,\ n-k \text{ units are in warm standby and each of them has failure rate } \lambda^* \end{cases}$$

The free period $\xi \sim$ Exp(λ_0). The busy period starts with a failure of some unit and terminates when all repair channels become empty. The entrance of $k(t)$ into state 0 are regeneration points. To carry out the analysis of this system and to estimate its reliability, we must evaluate, according to theorem 1, the value of p and μ^* or μ_0. We present below a theorem which helps to estimate these values.

Let μ be the mean service time in a channel, $\mu = E[\eta]$ and let $\bar{\lambda} = \max_{k=\overline{1,n}} \lambda_k$. Let $p = P$ (event A appears on a busy period).

Theorem 2. [1] If $\bar{\lambda}\mu \to 0$, then

$$P(\lambda_0 p\tau > t) \to e^{-t} \tag{3}$$

when $n \to \infty$.

Proof. Let us show that the condition of the theorem implies that $\lambda_0\mu_0 \to 0$, where $\mu_0 = E[$the length of the busy period$]$. Let us replace our system with another one which produces a Poisson flow with rate $\bar{\lambda}$ coming into one service channel. It is known from theory that the mean busy period is equal with $\hat{\mu}_0 = \mu/(1-\lambda\mu)$(see [4]). Cleary, the busy period of this new system is not shorter, on the average, than the busy period of the original system: $\hat{\mu}_0 \geq \mu_0$. Thus

$$\lambda_0\mu_0 \leq \lambda_0\hat{\mu}_0 \leq \bar{\lambda}\frac{\mu}{1-\bar{\lambda}\mu} \overset{n\to\infty}{\to} 0$$

According to the relationship (2), we have $P(\tau > t) \overset{n\to\infty}{\to} e^{-\lambda pt}$, thus:

$$P\left(\tau > \frac{1}{\lambda_0 p}t\right) \overset{n\to\infty}{\to} e^{-\lambda_0 p \frac{1}{\lambda_0 p}t}$$

The most interesting and difficult part in applying theorems 1 and 2 is the estimation of p, the probability of the appearance of the rare event. The key result is given by the following lemma proved by author.

Lemma 2. *If the service time η changes in such a way that*

$$\frac{E[\eta^{n+1}]}{(E[\eta])^n} \overset{n\to\infty}{\longrightarrow} 0$$

which satisfys the conditions $x^n\bar{G}(x) \to 0$ and $\int\limits_x^\infty \bar{G}(t)\mathrm{d}t \to 0$ if $x \to \infty$, then

$$p \approx p_0^* = \mathfrak{F}_{n,r} \cdot \prod_{i=1}^{n} \lambda_i \tag{4}$$

where

$$\mathfrak{F}_{n,r} = \int_0^\infty \frac{x^{n-r}}{(n-r)!} \left[\int_x^\infty \bar{G}(t)dt \right]^{r-1} \frac{\bar{G}(x)}{(r-1)!} \, \mathrm{d}x \tag{5}$$

with $\bar{G}(x) = 1 - G(x)$.

Proof. We present the main idea of this proof. The event

$$A = \{\text{system failure on a busy period}\}$$

can be split into events A_k where

$$A_k = \{\text{exactly } k \text{ units have been repaired before system failure occurred}\}, k \geq 0.$$

Denote $p_k = P(A_k)$. Thus:

$$p = p_0^* + p_1 + p_2 + ... = p_0^* + \tilde{p}$$

with $\tilde{p} = \sum\limits_{k=1}^{\infty} p_k$. The event A_0 corresponds to a monotone trajectory of $k(t)$: $n+1$ failures have accumulated without a single unit being released from the repair.

A remarkable fact is that

$$\tilde{p} = \sum_{k=1}^{\infty} p_k = O(p_0^*) \tag{6}$$

which means that $p \approx p_0^*$. If (6) is established, then the value of p_0^* can serve as an accurate estimate of p. This fact shows easily.

For the important cases $r = n$ and $r = 1$, the expression for p_0^* has a simple form:

$$p_0^* = \begin{cases} \frac{\lambda_1 \lambda_2 ... \lambda_n (E[\eta])^n}{n!} & \text{if} \quad r = n \\ \frac{\lambda_1 \lambda_2 ... \lambda_n E[\eta^n]}{n!} & \text{if} \quad r = 1 \end{cases} \tag{7}$$

Indeed by (5) for $r = 1$, we obtain:

$$\mathcal{F}_{n,1} = \int\limits_0^\infty \frac{x^{n-1}}{(n-1)!}\overline{G}(x)dx =$$

$$= \int\limits_0^\infty \overline{G}(x)\left(\frac{x^n}{n!}\right)' dx =$$

$$= \frac{x^n}{n!}\bar{G}(x)\mid_0^\infty - \int\limits_0^\infty \frac{x^n}{n!}g(x)dx =$$

$$= \frac{E[\eta^n]}{n!}$$

with the notation $g(x) = \overline{G}'(x)$.

For the case $r = n$, we then have:

$$\begin{aligned}
\mathcal{F}_{n,n} &= \frac{1}{(n-1)!}\int\limits_0^\infty\left[\int\limits_x^\infty \bar{G}(t)\mathrm{d}t\right]^{n-1}\bar{G}(x)\,\mathrm{d}x \\
&= -\frac{1}{(n-1)!}\int\limits_0^\infty\left[\int\limits_x^\infty \bar{G}(t)\mathrm{d}t\right]^{n-1}\mathrm{d}\left(\int\limits_x^\infty \bar{G}(t)\mathrm{d}t\right) \\
&= -\frac{1}{(n-1)!}\left[\int\limits_x^\infty \bar{G}(t)\mathrm{d}t\right]^{n-1}\int\limits_x^\infty \bar{G}(t)\mathrm{d}t\mid_0^\infty +
\end{aligned}$$

$$\begin{aligned}
+ &\quad \frac{1}{(n-2)!}\int\limits_0^\infty\int\limits_x^\infty \bar{G}(t)\mathrm{d}t\cdot\left(\int\limits_x^\infty \bar{G}(t)\mathrm{d}t\right)^{n-2}\bar{G}(t)\mathrm{d}t \\
= &\quad \frac{1}{(n-1)!}\left[\int\limits_0^\infty \bar{G}(t)\mathrm{d}t\right]^{n} + \frac{1}{(n-2)!}\int\limits_0^\infty\left(\int\limits_x^\infty \bar{G}(t)\mathrm{d}t\right)^{n-1}\bar{G}(t)\mathrm{d}t \\
= &\quad \frac{1}{(n-1)!}\left(E[\eta]\right)^n
\end{aligned}$$

2.1 Example

Assume that $\lambda_k = \lambda = \bar{\lambda} = 1$, $n = 3$, $r = 2$ and the repair time $\eta \sim \mathcal{U}(0, 0.04)$. Here $E[\eta] = \frac{0+0.04}{2} = 0.02$ and $E[\eta^4] = \frac{(0+0.04)^4}{4+1} = \frac{(0.04)^4}{5}$. Then

$$\frac{E[\eta^{n+1}]}{(E[\eta])^n} = \frac{E[\eta^4]}{(E[\eta])^3} = \frac{(0.04)^4}{5 \cdot (0.02)^3} = 0.064$$

which we consider as a small quantity.
We would like to estimate the probability of failure-free operation during $t = 20$. By lemma 2,

$$p_0^* = \lambda^3 \mathfrak{F}_{3,2} = 0.025 \cdot (0.04)^3 \cdot 0.06 = 4.26 \cdot 10^{-6}$$

We set $p \approx p_0^*$. The mean the busy period is

$$\mu_0 = \frac{E[\eta]}{1 - \lambda E[\eta]} = 0.0204$$

By (2) the bound on $P(\tau > t)$ in this case is:

$$e^{-4.2(6)\cdot 10^{-6}\cdot 20} \leq P(\tau > 20) \leq e^{-4.2(6)\cdot 10^{-6}\cdot 20} + 0.0204 \Leftrightarrow$$

$$0.918 \leq P(\tau > 20) \leq 0.938$$

Remark 1. *Does not apply the* (7) *because in this case* $r = 2 \neq n = 3$.

Remark 2. *We obtain one framing for the reliability of the system lifetime in certain case. In [5], theorem 2.1, establisch deviation from standard exponential distribution of the distribution of normalized system lifetime, that is*

$$\sup_{x \geq 0} \left| P\left(\frac{\tau}{E(\tau)} > x\right) - e^{-x} \right| < \frac{1 - \sqrt{1 - 4a_2}}{1 + \sqrt{1 - 4a_2}} \tag{8}$$

The right side to be approximated with a_2*, if* $a_2 \to 0$*. The remark 2.5. in [5] clarify the probabilistic meaning of the quantity* a_2 *at the relation*

$a_2 = 1 - \frac{E(\tau^2)}{2E^2(\tau)}$.

The relation (8) supply about one framing for the reliability of the system lifetime in the case while the system has little components (because the calculation of means lifetime is difficult).

In other side, in [6], the relation

$$\sup_{t \geq 0} \left| P\left(\sum_{i=1}^{N} \tau > t\right) - e^{-tp/\mu} \right| \leq \frac{E[\tau]p}{(E[\tau])^2} \tag{9}$$

provides that the cumulative distribution function of system lifetime τ *is rather close to* $1 - e^{\frac{p}{\mu}x}$*, that is the exponential distribution with the parameter* $\frac{p}{\mu}$.

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