General Mathematics Vol. 14, No. 2 (2006), 109–119

Optimal Quadrature Formulas in the Sense of Nikolski [1]

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Abstract

In this paper we will perform a method to obtain a quadrature formulas and we will study the optimality in sense of Nikolski for this quadrature formulas.

2000 Mathematics Subject Classification: 26D15, 65D30

Key words and phrases: quadrature rule, numerical integration, error bounds, optimal quadrature

1 Introduction

Let H be a linear space of real valued functions , defined and integrable on a finite interval $[a,b]\subset\mathbb{R}$ and $S:H\to\mathbb{R}$ be the integration operator defined by

$$S[f]=\int_a^b f(x)dx\,.$$

[1] *Received June 4, 2006*

Accepted for publication (in revised form) July 7, 2006

Let

$$\Lambda = \left\{ \lambda_i \mid \lambda_i : H \to \mathbb{R}, i = \overline{1,n} \right\}$$

be a set of linear functionals. For $f \in H$, one considers the quadrature formula

$$S[f] = Q_n[f] + \mathcal{R}_n[f] \tag{1}$$

where

$$Q_n[f] = \sum_{i=1}^{n} A_i \lambda_i(f)$$

and $\mathcal{R}_n[f]$ denotes the remainder term.

Remark 1.1. *Usually , $\lambda_i(f), i = \overline{1,n}$ are the values of the function f or of certain of its derivatives on the quadrature nodes from $[a,b]$.*

Definition 1.1. *The quadrature formula (1) is called optimal in the sense of Nikolski in the space H , if*

$$F_n(H, A, X) = \sup_{f \in H} |\mathcal{R}_n[f]| \ ,$$

attains the minimum value with regard to A and X , where $A = (A_1, \ldots, A_n)$ are the coefficients and $X = (x_1, \ldots, x_n)$ are the quadrature nodes.

2 Main Results

Let $(\Delta_m)_{m \in \mathbb{N}}$ be a division of $[a,b]$,

$$\Delta_m : a = x_0 < x_1 < x_2 < \cdots < x_{m-1} < x_m = b$$

and $(\xi_i)_{i=\overline{1,m}}$ a system of intermediate points ,

$$\xi_1 < \xi_2 < \cdots < \xi_m \, , \; \xi_i \in [x_{i-1}, x_i] \, \text{for } i = \overline{1,m} \, .$$

Theorem 2.1. *If $f \in C^{n-1}[a,b]$, $f^{(n-1)}$ is absolutely continous , then*

$$\int_a^b f(t)dt = \sum_{k=0}^{n-1}\sum_{i=0}^{m} A_{k,i} f^{(k)}(x_i) + \mathcal{R}_n[f] \tag{2}$$

where

$$\mathcal{R}_n[f] = (-1)^n \int_a^b K_n(t) f^{(n)}(t)dt \tag{3}$$

$$K_n(t) = \begin{cases} \dfrac{(t-\xi_i)^n}{n!}, & t \in [x_{i-1}\,,\, x_i)\ ,\ i = \overline{1, m-1} \\ \dfrac{(t-\xi_m)^n}{n!}, & t \in [x_{m-1}, b] \end{cases} \tag{4}$$

and for $k = \overline{0, n-1}$

$$A_{k,i} = (-1)^k \frac{(x_i-\xi_i)^{k+1} - (x_i - \xi_{i+1})^{k+1}}{(k+1)!}\ ,\ i = \overline{1, m-1}$$
$$A_{k,0} = (-1)^{k+1} \frac{(a-\xi_1)^{k+1}}{(k+1)!}$$
$$A_{k,m} = (-1)^k \frac{(b-\xi_m)^{k+1}}{(k+1)!}$$

Proof. We prove (2) by induction. For $n = 1$ we have

$$-\int_a^b K_1(t) f'(t)dt = -\left[\sum_{i=1}^{m-1}\int_{x_{i-1}}^{x_i}(t-\xi_i)f'(t)dt + \int_{x_{m-1}}^{b}(t-\xi_m)f'(t)dt\right] =$$

$$= -\left[\sum_{i=1}^{m-1}(t-\xi_i)f(t)\Big|_{x_{i-1}}^{x_i} + (t-\xi_m)f(t)\Big|_{x_{m-1}}^{b} - \int_a^b f(t)dt\right] =$$

$$= -\left[\sum_{i=1}^{m-1}(x_i-\xi_i)f(x_i) - \right.$$

$$\left. - \sum_{i=1}^{m-1}(x_{i-1}-\xi_i)f(x_{i-1}) + (b-\xi_m)f(b) - (x_{m-1}-\xi_m)f(x_{m-1})\right] + \int_a^b f(t)dt =$$

$$= -\left[\sum_{i=1}^{m-1}\left((x_i-\xi_i) - (x_i-\xi_{i+1})\right)f(x_i) - (a-\xi_1)f(a) + (b-\xi_m)f(b)\right]$$

$$+\int_a^b f(t)dt = -\sum_{i=0}^{m} A_{0,i} f(x_i) + \int_a^b f(t)dt\,.$$

Therefore

$$\int_a^b f(t)dt = \sum_{i=0}^{m} A_{0,i} f(x_i) - \int_a^b K_1(t) f'(t)dt$$

For $n = 2$ we have

$$\int_a^b K_2(t) f''(t)dt = \sum_{i=1}^{m-1} \int_{x_{i-1}}^{x_i} \frac{(t-\xi_i)^2}{2!} f''(t)dt + \int_{x_{m-1}}^{b} \frac{(t-\xi_m)^2}{2!} f''(t)dt$$

$$= \sum_{i=1}^{m-1} \frac{(t-\xi_i)^2}{2!} f'(t)\Big|_{x_{i-1}}^{x_i} + \frac{(t-\xi_m)^2}{2!} f'(t)\Big|_{x_{m-1}}^{b} - \int_a^b K_1(t) f'(t)dt$$

$$= \sum_{i=1}^{m-1} \frac{(x_i-\xi_i)^2}{2!} f'(x_i) - \sum_{i=1}^{m-1} \frac{(x_{i-1}-\xi_i)^2}{2!} f'(x_{i-1}) +$$

$$+ \frac{(b-\xi_m)^2}{2!} f'(b) - \frac{(x_{m-1}-\xi_m)^2}{2!} f'(x_{m-1}) -$$

$$-\int_a^b K_1(t) f'(t)dt = \sum_{i=1}^{m-1} \frac{(x_i-\xi_i)^2 - (x_i-\xi_{i+1})^2}{2!} f'(x_i) -$$

$$- \frac{(a-\xi_1)^2}{2!} f'(a) + \frac{(b-\xi_m)^2}{2!} f'(b) -$$

$$-\int_a^b K_1(t) f'(t)dt = -\sum_{i=0}^{m} A_{1,i} f'(x_i) - \sum_{i=0}^{m} A_{0,i} f(x_i) + \int_a^b f(t)dt\,.$$

Therefore

$$\int_a^b f(t)dt = \sum_{k=0}^{1} \sum_{i=0}^{m} A_{k,i} f^{(k)}(x_i) + \int_a^b K_2(t) f''(t)dt\,.$$

Now suppose that (2) holds for an arbitrary n. We have to prove that (3) holds for $n \to n+1$. We have

$$(-1)^{n+1} \int_a^b K_{n+1}(t) f^{(n+1)}(t)dt = (-1)^{n+1} \left[\sum_{i=1}^{m-1} \int_{x_{i-1}}^{x_i} \frac{(t-\xi_i)^{n+1}}{(n+1)!} f^{(n+1)}(t)dt + \right.$$

$$+\int_{x_{m-1}}^{b}\frac{(t-\xi_m)^{n+1}}{(n+1)!}f^{(n+1)}(t)dt\Bigg]=$$

$$=(-1)^{n+1}\left[\sum_{i=1}^{m-1}\frac{(t-\xi_i)^{n+1}}{(n+1)!}f^{(n)}(t)\Big|_{x_{i-1}}^{x_i}+\right.$$

$$\left.+\frac{(t-\xi_m)^{n+1}}{(n+1)!}f^{(n)}(t)\Big|_{x_{m-1}}^{b}-\int_a^b K_n(t)f^{(n)}(t)dt\right]=$$

$$=(-1)^{n+1}\left[\sum_{i=1}^{m-1}\frac{(x_i-\xi_i)^{n+1}}{(n+1)!}f^{(n)}(x_i)-\sum_{i=1}^{m-1}\frac{(x_{i-1}-\xi_i)^{n+1}}{(n+1)!}f^{(n)}(x_{i-1})+\right.$$

$$+\frac{(b-\xi_m)^{n+1}}{(n+1)!}f^{(n)}(b)-$$

$$\left.-\frac{(x_{m-1}-\xi_m)^{n+1}}{(n+1)!}f^{(n)}(x_{m-1})\right]+(-1)^n\int_a^b K_n(t)f^{(n)}(t)dt=$$

$$=(-1)^{n+1}\sum_{i=1}^{m-1}\frac{(x_i-\xi_i)^{n+1}-(x_i-\xi_{i+1})^{n+1}}{(n+1)!}f^{(n)}(x_i)+(-1)^n\frac{(a-\xi_1)^{n+1}}{(n+1)!}f^{(n)}(a)+$$

$$+(-1)^{n+1}\frac{(b-\xi_m)^{n+1}}{(n+1)!}f^{(n)}(b)+(-1)^n\int_a^b K_n(t)f^{(n)}(t)dt=$$

$$=-\sum_{i=0}^{m}A_{n,i}f^{(n)}(x_i)-\sum_{k=0}^{n-1}\sum_{i=0}^{m}A_{k,i}f^{(k)}(x_i)+\int_a^b f(t)dt\,.$$

Therefore

$$\int_a^b f(t)dt=\sum_{k=0}^{n}\sum_{i=0}^{m}A_{k,i}f^{(k)}(x_i)+(-1)^{n+1}\int_a^b K_{n+1}(t)f^{(n+1)}(t)dt\,.$$

Remark 2.1. *The quadrature formulas of type (2) with equidistant knots had obtain from this method in [1], [2], [3], [5], [6], [7], [9], [10].*

Remark 2.2. *If* $\xi_1=a$ *and* $\xi_m=b$ *then quadrature formula (2) is open type.*

Next, we will study the optimality in sense of Nikolski for this quadrature formulas. Let $H^{n,p}[a,b]=\left\{f:[a,b]\to\mathbb{R}\Big| f\in C^{n-1}[a,b]\,,\ f^{(n)}\in L^p[a,b]\right\}$. If $f\in H^{n,p}[a,b]$ for rest term we have the evaluation

$$|\mathcal{R}_n[f]|\leq\left[M_n^{[p]}[f]\right]^{\frac{1}{p}}\left[\int_a^b|K_n(t)|^q\,dt\right]^{\frac{1}{q}} \tag{5}$$

where

$$M_n^{[p]}[f]=\int_a^b\left|f^{(n)}(t)\right|^p dt\,,\ \frac{1}{p}+\frac{1}{q}=1$$

with remark that in cases $p=1$ and $p=\infty$ this evaluation is

$$|\mathcal{R}_n[f]|\leq M_n^{[1]}[f]\sup_{t\in[a,b]}|K_n(t)| \tag{6}$$

$$|\mathcal{R}_n[f]|\leq M_n^{\infty}[f]\int_a^b|K_n(t)|\,dt \tag{7}$$

where

$$M_n^{[1]}[f]=\int_a^b\left|f^{(n)}(t)\right|dt$$
$$M_n^{\infty}[f]=\sup_{t\in[a,b]}\left|f^{(n)}(t)\right|\,.$$

The quadrature formula (2) is optimal in the sense of Nikolski in $H^{n,p}[a,b]$, if

$$\int_a^b|K_n(t)|^q\,dt\,,\ \frac{1}{p}+\frac{1}{q}=1$$

attains the minimum value.

Theorem 2.2. *If $f\in H^{n,p}[a,b]$, $p>1$,then quadrature formula of the form (2), optimal with regard to the error, is*

$$\int_a^b f(x)dx=\sum_{k=0}^{n-1}\sum_{i=0}^{m}A_{k,i}^*f^{(k)}(x_i^*)+\mathcal{R}_n^*[f]$$

where , for $k = \overline{0, n-1}$

$$A^*_{k,0} = \frac{(b-a)^{k+1}}{2^{k+1} m^{k+1} (k+1)!}$$
$$A^*_{k,i} = [1 + (-1)^k] \frac{(b-a)^{k+1}}{2^{k+1} m^{k+1} (k+1)!}\,, i = \overline{1, m-1}$$
$$A^*_{k,m} = (-1)^k \frac{(b-a)^{k+1}}{2^{k+1} m^{k+1} (k+1)!}$$
$$x^*_i = a + \frac{b-a}{m} i\,,\ i = \overline{1, m-1}$$

with

$$|\mathcal{R}^*_n[f]| \le \frac{(b-a)^n}{(2m)^n n!} \cdot \left(\frac{b-a}{qn+1} \right)^{\frac{1}{q}} \cdot \left[M_n^{[p]}[f] \right]^{\frac{1}{p}}\,.$$

Proof. We will determine the parameters A and X for which

$$F(A, X) = \int_a^b |K_n(t)|^q \, dt$$

attains the minimum value . We have

$$F(A, X) = \sum_{i=1}^{m-1} \int_{x_{i-1}}^{x_i} \left| \frac{(t-\xi_i)^n}{n!} \right|^q dt + \int_{x_{m-1}}^{b} \left| \frac{(t-\xi_m)^n}{n!} \right|^q dt$$

$$= \frac{1}{(qn+1)(n!)^q} \left[\sum_{i=1}^{m} (x_i - \xi_i)^{qn+1} + \sum_{i=1}^{m} (\xi_i - x_{i-1})^{qn+1} \right]\,.$$

The optimal nodes constitute the solution of the system

$$(8)\begin{cases} \dfrac{\partial F(A,X)}{\partial x_k} = \dfrac{1}{(n!)^q} \left[(x_k - \xi_k)^{qn} - (\xi_{k+1} - x_k)^{qn} \right] = 0\,,\ k = \overline{1, m-1} \\ \dfrac{\partial F(A,X)}{\partial \xi_k} = \dfrac{1}{(n!)^q} \left[-(x_k - \xi_k)^{qn} + (\xi_k - x_{k-1})^{qn} \right] = 0\,,\ k = \overline{1, m} \end{cases}$$

From (8) we obtain

$$(9) \qquad \xi_k = \frac{x_{k-1} + x_k}{2}\,,\ k = \overline{1, m}$$

$$(10) \qquad x_{k+1} - 2x_k + x_{k-1} = 0\,,\ k = \overline{1, m-1}\,.$$

From recurrent relation (10) we obtain

$$x_k = a + \frac{b-a}{m}k\,,\ \ k = \overline{1, m-1} \tag{11}$$

From (9) and(11) follows that

$$\begin{aligned}
A_{k,0} &= \frac{(b-a)^{k+1}}{2^{k+1}m^{k+1}(k+1)!}\\
A_{k,i} &= [1+(-1)^k]\frac{(b-a)^{k+1}}{2^{k+1}m^{k+1}(k+1)!}\,, i = \overline{1, m-1}\\
A_{k,m} &= (-1)^k\frac{(b-a)^{k+1}}{2^{k+1}m^{k+1}(k+1)!}\,.
\end{aligned}$$

Because the quadratic form

$$\phi = \sum_{i=1}^{m-1}\sum_{j=1}^{m}\frac{\partial^2 F(A,X)}{\partial x_i \partial \xi_j}a_i b_j + \sum_{j=1}^{m}\sum_{i=1}^{m-1}\frac{\partial^2 F(A,X)}{\partial \xi_j \partial x_i}b_j a_i$$

$$+\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\frac{\partial^2 F(A,X)}{\partial x_i x_j}a_i a_j + \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^2 F(A,X)}{\partial \xi_i \partial \xi_j}b_i b_j$$

in cross point (A, X) is positive, namely

$$\phi = \frac{qn}{(n!)^q}\cdot\left(\frac{b-a}{2m}\right)^{qn-1}\cdot\left\{\sum_{i=1}^{m-1}\left[(a_i-b_i)^2+(a_i-b_{i+1})^2\right]+b_1^2+b_m^2\right\}$$

then $F(A, X)$ attains the minimum value for the knots $X^* = (x_i^*)_{i=\overline{1,m-1}}$ and coefficients $A^* = (A^*_{k,i})_{k=0}^{n-1}{}_{i=0}^{m}$, where

$$\begin{aligned}
x_i^* &= a + \frac{b-a}{m}i\,,\ i = \overline{1, m-1}\\
A^*_{k,0} &= \frac{(b-a)^{k+1}}{2^{k+1}m^{k+1}(k+1)!}\\
A^*_{k,i} &= [1+(-1)^k]\frac{(b-a)^{k+1}}{2^{k+1}m^{k+1}(k+1)!}\,, i = \overline{1, m-1}\\
A^*_{k,m} &= (-1)^k\frac{(b-a)^{k+1}}{2^{k+1}m^{k+1}(k+1)!}
\end{aligned}$$

Finally, we have

$$F(A^*, X^*) = \min_{A,X} F(A, X) = \frac{(b-a)^{qn+1}}{(qn+1)(n!)^q(2m)^{qn}}\,,$$

and

$$|\mathcal{R}_n^*[f]| \leq \frac{(b-a)^n}{(2m)^n n!} \cdot \left(\frac{b-a}{qn+1}\right)^{\frac{1}{q}} \cdot \left[M_n^{[p]}[f]\right]^{\frac{1}{p}}\,.$$

In this way we prove follow result:

Theorem 2.3. *The quadrature formula of the form (2) is optimal in the sense of Nikolski for $p = \infty$ if it has the coefficients and knots*

$$A_{k,0}^* = \frac{(b-a)^{k+1}}{2^{k+1}m^{k+1}(k+1)!}$$

$$A_{k,i}^* = [1+(-1)^k]\frac{(b-a)^{k+1}}{2^{k+1}m^{k+1}(k+1)!}\,, i = \overline{1, m-1}$$

$$A_{k,m}^* = (-1)^k \frac{(b-a)^{k+1}}{2^{k+1}m^{k+1}(k+1)!}$$

$$x_i^* = a + \frac{b-a}{m} i\,,\ i = \overline{1, m-1}$$

and there is for rest term evaluation

$$|\mathcal{R}_n^*[f]| \leq \frac{(b-a)^{n+1}}{(n+1)!(2m)^n} \cdot M_n^{\infty}[f]\,.$$

The optimal quadrature formulas had obtain by S.M. Nikolski (see [8]).In [4] T. Cătinaş and G. Coman obtain the optimal quadrature formulas using φ- function method.

For example if $f \in H^{1,2}[0,1]$ then quadrature formula of the form (2), optimal with regard to the error, is

$$\int_0^1 f(x)dx = \frac{1}{2m} \cdot \left[f(0) + 2\sum_{i=1}^{m-1} f\left(\frac{i}{m}\right) + f(1)\right] + \mathcal{R}_1^*[f]\,,$$

where

$$|\mathcal{R}_1^*[f]| \leq \frac{1}{2m\sqrt{3}}\,\|f'\|_2\,.$$

For $f \in H^{2,2}[0,1]$ we have

$$\int_0^1 f(x)dx = \frac{1}{2m}\cdot\left[f(0)+2\sum_{i=1}^{m-1} f\left(\frac{i}{m}\right)+f(1)\right]+\frac{1}{8m^2}f'(0)-\frac{1}{8m^2}f'(1)+\mathcal{R}_2^*[f]\,,$$

where

$$|\mathcal{R}_2^*[f]| \le \frac{1}{8m^2\sqrt{5}}\left\|f''\right\|_2 .$$

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