About Asymptotic Behaviour of Solutions of Differential Equations as $x \to \infty$ [1]

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Abstract

The aim of this paper is the study of one additional problem regarding the theory of differential equations.

2000 Mathematics Subject Classification: 34K15, 47H10

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Assume that we have a differential equation

(1)
$$y'' + q(x)y = 0$$

and let $q(x)$, as $x \to +\infty$, have a positive limit which we can assume to be unity without a loss of generality. Then

$$q(x) = 1 + \alpha(x),$$

where $\alpha(x) \to 0$ $(x \to \infty)$, and equation (1) takes the form

(2) $$y'' + y + +\alpha(x)y = 0.$$

As $x \to +\infty$ we obtain a "limiting" equation

(3) $$y'' + y = 0$$

whose all solutions $y = A \cos x + B \sin x$ are bounded for $x \in \mathbb{R}$. Therefore, it is natural to expect that the solutions of the differential equations (2) are also bounded for $x \to +\infty$.

Theorem 1.1. (see [2]) *If $\alpha(x)$ is a continuously differentiable function, and*

(4) $$|\alpha(x)| < \frac{a}{x}, \qquad |\alpha'(x)| < \frac{a}{x^2}$$

for all sufficiently large x, where a is a positive constant, then every solution of the differential equation (2) is bounded for $x \to +\infty$.

Proof. We multiply all the terms of equation (2) by y' and integrate the result with respect to x from a certain positive number x_0, which will be chosen in the requisite way later on, to x:

$$\left(y'^2\right)\Big|_{x=x_0}^{x=x} + \left(y^2\right)\Big|_{x=x_0}^{x=x} + 2 \int_{x_0}^{x} \alpha(x)yy'\mathrm{d}x = 0.$$

Integrating the last term on the left-hand side by parts, we obtain

$$y'^2(x) - y'^2(x_0) + y^2(x) - y^2(x_0) + (\alpha y^2)\Big|_{x=x_0}^{x=x} - \int_{x_0}^{x} \alpha'(x)y^2(x)\mathrm{d}x = 0,$$

whence

(5) $$y^2(x) \le y'^2(x) + y^2(x) \le c(x_0) + |\alpha(x)|y^2(x) + \int_{x_0}^{x} |\alpha'(x)|\, y^2(x)\mathrm{d}x$$

where $C(x_0) \ge 0$ is an expression depending only on x_0.

We denote by M the greatest value of the function $|y(x)|$ on the interval $[x_0, x]$. Suppose it is attained at a certain point $\xi \in [x_0, x]$. Using inequalities (4) and (5), we obtain

$$M^2 \le C(x_0) + \frac{M^2 a}{\xi} + M^2 a \left(\frac{1}{x_0} - \frac{1}{\xi}\right), \qquad M^2 \left(1 - \frac{a}{x}\right) \le C(x_0).$$

If we choose $x_0 \geq 2a$, we get

$$M^2 \leq 2C(x_0),$$

and this proves the assertion since the quantity $2C(x_0)$ does not depend on x.

Application (see [1]). Show that all solutions of the equation

$$y'' + \left(1 + e^{-x^2} - \frac{1}{x+2}\right) y = 0$$

are bounded on $[0, +\infty)$.

If we impose stronger conditions on decrease on $\alpha(x)$

(6)
$$\alpha(x) = O\left(\frac{1}{x^2}\right), \qquad x \to +\infty,$$

then the greater closeness of equation (2) to the limiting equation (3) will entail not only the boundedness of the solutions but also their asymptotic approximation to trigonometric functions, i.e. the solutions of the limiting equation. We can show that in this case for every solution $y(x)$ of equation (2) there holds an asymptotic formula

$$y(x) = A \sin(x + \delta_0) + O\left(\frac{1}{x}\right),$$

where A and δ_0 are some constants.

Thus the equation

$$y'' + \left(A - \frac{\nu^2 - 1/4}{x^2}\right) y = 0,$$

where $\alpha(x) = \left(\frac{1}{4} - \nu^2\right)/x^2$ satisfies condition (6). The solution of that equation is connected with Bessel's function $J_\nu(x)$ by the relation $y(x) = \sqrt{x} J_\nu(x)$, which leads to an asymptotic formula for Besel functions:

$$J_\nu(x) = \frac{A}{\sqrt{x}} \sin(x + \delta_0) + O\left(\frac{1}{x^{3/2}}\right),$$

where $A = \sqrt{\dfrac{2}{\pi}}$ and $\delta_0 = -\nu\dfrac{\pi}{2} + \dfrac{\pi}{4}$.

Examples given below show that the asymptotic behaviour of the solutions of a differential equation cannot always be deduced from the behaviour

of the solutions of a limiting equation.

By way an example, we consider two equations (see [1]):

(7)
$$y'' - \frac{2}{x}y' + y = 0,$$

(8)
$$y'' + \frac{2}{x}y' + y = 0.$$

When $x \to +\infty$, the limiting equation for them is

(9)
$$y'' + y = 0.$$

All the solutions of the equation (9) are bounded on $[1, +\infty)$. Equation (7) has a fundamental system of solutions

$$y_1(x) = \frac{\sin x}{x}, \qquad y_2(x) = \frac{\cos x}{x};$$

consequently, its all solutions are bounded on $[1, +\infty)$ and even tend to zero as $x \to +\infty$.

References

[1] Mioara Boncuţ, Amelia Bucur, *Capitole de matematici speciale*, Ed. Alma Mater, Sibiu, 2001 (in Romanian).

[2] M.L. Krasnov, *Ordinary Differential Equations*, Mir Publishers, Moscow, 1987.

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