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On the Univalence of Some Integral Operators [1]

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Abstract

In this paper some integral operators are studied and are determined conditions for the univalence of these operators.

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1 Introduction

Let A be the class of the functions f which are regular in the unit disc $U = \{z \in C, |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$. We denote by S the class of the functions $f \in A$ which are univalent in U.

L.V. Ahlfors [1] and J. Becker [2] had obtained the next univalence criterion.

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Theorem A. *Let c be a complex number, $|c| \leq 1, c \neq -1$. If $f(z) = z + a_2z^2 + \ldots$ is a regular function in U and*

$$\left| c|z|^2 + \left(1 - |z|^2\right) \frac{zf''(z)}{f'(z)} \right| \leq 1, \tag{1}$$

for all $z \in U$, then the function $f(z)$ is regular and univalent in U.

Further, V. Pescar [6] gave

Theorem B. *Let α be a complex number, $Re\alpha > 0$, and c a complex number, $|c| \leq 1$, $c \neq -1$ and $f(z) = z + \cdots$ a regular function in U. If*

$$\left| c|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1, \tag{2}$$

for all $z \in U$, then the function

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u)du\right]^{\frac{1}{\alpha}} = z + \ldots, \tag{3}$$

is regular and univalent in U.

In this paper we will need the following theorems.

Theorem C. [5] *Let $f \in A$ satisfy the condition*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1, \; z \in U, \tag{4}$$

then f is univalent in U.

Theorem D. [8] *Let α be a complex number, $Re\alpha > 0$ and c a complex number, $|c| \leq 1$, $c \neq -1$ and $f \in A$. If*

$$\frac{1 - |z|^{2Re\alpha}}{Re\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - |c|, \tag{5}$$

for all $z \in U$, then for any complex number β, $Re\beta \geq Re\alpha$, the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} \, f'(u)du\right]^{\frac{1}{\beta}} \tag{6}$$

is in the class S.

Schwarz Lemma. **[3]** *Let* $f(z)$ *the function regular in the disk* $U_R = \{z \in C : \ |z| < R\}$, *with* $|f(z)| < M, M$ *fixed. If* $f(z)$ *has in* $z = 0$ *one zero with multiply* $\geq m$, *then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m,\ z \in U_R \tag{7}$$

the equality (in (7) for $z \neq 0$*) can hold only if* $f(z) = e^{i\theta}\frac{M}{R^m}z^m$, *where* θ *is constant.*

2 Main results

Theorem 1. *Let the function* $g \in A$ *satisfy* (4), M *be a positiv real number fixed and* c *be a complex number. If* $\alpha \in \left[\frac{2M+1}{2M+2}, \frac{2M+1}{2M}\right]$,

$$|c| \leq 1 - \left|\frac{\alpha - 1}{\alpha}\right|(2M+1),\ c \neq -1 \tag{8}$$

and

$$|g(z)| \leq M \tag{9}$$

for all $z \in U$, *then the function*

$$G_\alpha(z) = \left[\alpha \int_0^z [g(u)]^{\alpha-1} du\right]^{\frac{1}{\alpha}} \tag{10}$$

is in the class S.

Proof. From (10) we have

$$G_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1}\left(\frac{g(u)}{u}\right)^{\alpha-1} du\right]^{\frac{1}{\alpha}}. \tag{11}$$

Let us consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\alpha-1} du. \tag{12}$$

The function f is regular in U.

From (12) we get $f'(z) = \left(\frac{g(z)}{z}\right)^{\alpha-1}$, $f''(z) = (\alpha-1)\left(\frac{g(z)}{z}\right)^{\alpha-2} \frac{zg'(z)-g(z)}{z^2}$ and

$$\begin{aligned} &\left|c|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right)\frac{zf''(z)}{\alpha f'(z)}\right| = \\ &= \left|c|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right)\frac{\alpha-1}{\alpha}\left(\frac{zg'(z)}{g(z)} - 1\right)\right| \leq \\ &\leq |c| + \left|\frac{\alpha-1}{\alpha}\right|\left(\left|\frac{z^2g'(z)}{g^2(z)}\right|\frac{|g(z)|}{|z|} + 1\right) \end{aligned} \tag{13}$$

for all $z \in U$.

We have $g(0) = 0$ and $|g(z)| < M$ and by the Schwarz-Lemma we obtain $|g(z)| < M|z|$. Using (13), we have

$$\begin{aligned} &\left|c|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right)\frac{zf''(z)}{\alpha f'(z)}\right| \leq \\ &\leq |c| + \left|\frac{\alpha-1}{\alpha}\right|\left[\left(\left|\frac{z^2g'(z)}{g^2(z)} - 1\right| + 1\right)M + 1\right] \end{aligned} \tag{14}$$

From (14) and since g satisfies the condition (4) we have

$$\left|c|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right)\frac{zf''(z)}{\alpha f'(z)}\right| \leq |c| + \left|\frac{\alpha-1}{\alpha}\right|(2M+1) \tag{15}$$

For $\alpha \in \left[\frac{2M+1}{2M+2}, \frac{2M+1}{2M}\right]$ we have

$$|c| \leq 1 - \left|\frac{\alpha-1}{\alpha}\right|(2M+1) \leq 1 \tag{16}$$

and, hence, we get

$$\left|c|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right)\frac{zf''(z)}{\alpha f'(z)}\right| \leq 1, \; z \in U. \tag{17}$$

for all $z \in U$.

From (12) we have $f'(z) = \left(\frac{g(z)}{z}\right)^{\alpha-1}$ and by Theorem B for α real number, $\alpha > 0$, it results that the function G_α is in the class S.

Theorem 2. *Let $g \in A$, α be a real number, $\alpha \geq 1$, and c a complex number, $|c| \leq \frac{1}{\alpha}$, $c \neq -1$. If*

$$\left|\frac{g''(z)}{g'(z)}\right| \leq 1,\ z \in U \tag{18}$$

then the function

$$H_\alpha(z) = \{\alpha \int_0^z [ug'(u)]^{\alpha-1}\, du\}^{\frac{1}{\alpha}} \tag{19}$$

is in the class S.

Proof. We observe that

$$H_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1}\left(g'(u)\right)^{\alpha-1} du\right]^{\frac{1}{\alpha}} \tag{20}$$

Let us consider the function

$$p(z) = \int_0^z [g'(u)]^{\alpha-1}\, du. \tag{21}$$

The function p is regular in U.

From (21) we have

$$p'(z) = \left(g'(z)\right)^{\alpha-1},\ p''(z) = (\alpha-1)\left[g'(z)\right]^{\alpha-2} g''(z)$$

and we obtain

$$\left|c|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right)\frac{zp''(z)}{\alpha p'(z)}\right| = \left|c|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right)\frac{zg''(z)}{g'(z)}\frac{\alpha-1}{\alpha}\right|. \tag{22}$$

From (22), (18) and the conditions of theorem we get

$$\left| c|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\frac{zp''(z)}{\alpha p'(z)}\right)\right| \le |c| + \frac{\alpha - 1}{\alpha} \le 1 \tag{23}$$

for all $z \in U$.

By Theorem B for α real number, $\alpha \ge 1$, and since $p'(z) = [g'(z)]^{\alpha-1}$ it results that the function H_α is in the class S.

Theorem 3. *Let $g \in A$ satisfies (4), α be a complex number, $M > 1$ fixed, $Re\alpha > 0$ and c be a complex number, $|c| < 1$. If*

$$|g(z)| \le M \tag{24}$$

for all $z \in U$, then for any complex number β

$$Re\beta \ge Re\alpha \ge \frac{2M+1}{|\alpha|\,(1 - |c|)} \tag{25}$$

the function

$$H_\beta(z) = \left[\beta \int_0^z u^{\beta-1}\left(\frac{g(u)}{u}\right)^{\frac{1}{\alpha}} du\right]^{\frac{1}{\beta}}. \tag{26}$$

is in the class S.

Proof. Let us consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\frac{1}{\alpha}} du. \tag{27}$$

The function f is regular in U. From (27) we have:

$$f'(z) = \left(\frac{g(z)}{z}\right)^{\frac{1}{\alpha}}, \qquad f''(z) = \frac{1}{\alpha}\left(\frac{g(z)}{z}\right)^{\frac{1}{\alpha-1}} \frac{zg'(z) - g(z)}{z^2}$$

and

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{1-|z|^{2Re\alpha}}{|\alpha|Re\alpha}\left|\frac{zg'(z)}{g(z)}\right| + \frac{1-|z|^{2Re\alpha}}{|\alpha|Re\alpha} \tag{28}$$

for all $z \in U$ and hence, we have

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{1-|z|^{2Re\alpha}}{|\alpha|Re\alpha}\left(\left|\frac{z^2g'(z)}{g^2(z)}\right| \cdot \left|\frac{g(z)}{z}\right| + 1\right) \tag{29}$$

for all $z \in U$.

By the Schwarz-Lemma also $|g(z)| \le M|z|,\ z \in U$ and using (29) we obtain

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{1-|z|^{2Re\alpha}}{|\alpha|Re\alpha}\left(\left|\frac{z^2g'(z)}{g^2(z)} - 1\right| + 1\right)M + 1 \tag{30}$$

for all $z \in U$.

From (30) and since g satisfies the condition (4) we get

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{1-|z|^{2Re\alpha}}{Re\alpha}\ \frac{2M+1}{|\alpha|} \le \frac{2M+1}{|\alpha|Re\alpha} \tag{31}$$

for all $z \in U$.

From (25) we have $\frac{2M+1}{|\alpha|Re\alpha} \le 1-|c|$ and using (31) we obtain

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1-|c|. \tag{32}$$

From (27) we obtain $f'(z) = \left(\frac{g(z)}{z}\right)^{\frac{1}{\alpha}}$ and using (32) by Theorem D we conclude that the function H_β is in the class S.

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