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Cases of dissipativity in the Hoph parameter system [1]

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Abstract

Application for the control in the steel bars using the ultrasounds method

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1 The problem's demand

In the system of quality insurance AQ04 for the production of nuclear centrals equipment ,the control of primar materials receiving is essential.

Problem: A company receives steel bars of different diameters but of the same length for the production of "Screws couple Fideri"and we want to find out the possible fissures in their inside.The controll procedure that system AQ04 demands is the ultrasounds one(Cod ASME 82101) which

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consists in the covering of the surfaces with a thick substance (for ex: glycerine) and the emission of a sound with a frequency between 2-10 Mhz in the left part of the bar.The machine which measures and controlls is connected to a process computer and it has two sensors which can be moved up and down on the bar's surfaces.If there are not fissures ,the signal is linear and uniform otherwise it is interrupted.From a case study of the specialists we have the following parameters dynamical system :

$$\begin{cases} x' + \alpha \cdot x + y^2 + z^2 = 0 \\ y' + \gamma \cdot y - x \cdot y - \beta \cdot z = 0; \alpha, \beta, \gamma \in \Re; \alpha, \beta > 0 \\ z' + \gamma \cdot z - x \cdot z + \beta \cdot z = 0 \end{cases} \tag{1.1}$$

There are two main methods of symplifying the parameters dynamical system study.First is the "Method of normal forms" which has its roots in the licence paper of H.Poincare(1929).This method consists in finding a coordinates system in which the dynamical system has in a certain way the easiest form.The other one is the "Manifold method " introduced by E.Hopf(1970).In the situation demanded by our problem we are interested only in the dissypativity cases which means the ones with potential energy links.

2 Analytic study

2.1 The dynamical system is dissipative

It will also be sufficient for the proof of global solvability.Let us introduce a new unknown function $u = x + \alpha/2 + \gamma$.Then the system (1.1) can be rewritten in the form:

$$\begin{cases} u' + \alpha \cdot u + y^2 + z^2 = \alpha(\alpha/2 - \gamma) \\ y' + 1/2 \cdot \alpha \cdot y - u \cdot y - \beta \cdot z = 0 \\ z' + 1/2 \cdot \alpha \cdot z - z \cdot u + \beta \cdot y = 0 \end{cases} \tag{1.2}$$

These equations implies that :

$$\frac{d}{dt}(u^2 + y^2 + z^2) + \alpha \cdot (u^2 + y^2 + z^2) = 2 \cdot \alpha \cdot u(\alpha/2 - \gamma) - u^2 \cdot \alpha$$

on any interval of existence of solutions.Hence,

$$\tfrac{d}{dt}(u^2 + y^2 + z^2) + \alpha \cdot (u^2 + y^2 + z^2) \leq \alpha(\alpha/2 - \gamma)^2$$

Using **Gronwall's lema** (continuous case) we obtain:

$$u(t)^2 + y(t)^2 + z(t)^2 \leq$$
$$\exp(-\alpha \cdot t) \cdot (u(0)^2 + y(0)^2 + z(0)^2) + (\alpha/2 - \gamma)^2 (1 - \exp(-\alpha \cdot t)).$$

Firstly ,this equations enables us to prove the global solvability of problem (1.1)for any initial condition and, secondly,it means that the set:

$$B_0 = \{(u, y, z) : (u + \alpha/2 - \gamma)^2 + y^2 + z^2 \leq 1 + (\alpha/2 - \gamma)^2\}$$

is absorbing for dynamical system generated by the Cauchy problems for (1.1).It is also a connected compact set in $\Re^3$.

Because $\alpha > 0$ then $\exp(-\alpha \cdot t) \to \infty$ when t $\to \infty$ implies that $S_t(.) = \exp(-t(.))$ is evolution semigroup of dynamical dissipative system(1.1) and:

$$\lim(u(t)^2 + y(t)^2 + z(t)^2) = (\alpha/2 - \gamma).$$

Thus the set B_0 is a positively invariant set for $(\Re^3, S_t)$,that means $S_t(B_0) \subseteq B_0$ for every $\nu \in B_0$.This is obviously because $\exp(-t)(\nu) \leq \nu$.Further more (1.1) is generalised dissipative dynamical system because for any $v = (u, y, z)$

$(f(v), v) = \alpha/2(\|v\|^2 + (u - \alpha/2 + \gamma)^2 - (\alpha/2 - \gamma)^2) < 0$ if only if $\|v\| > \gamma - \alpha/2$.

2.2 The structure of global attractor

In order to describe the structure of the global attractor A ,the following assertions is the main result.

Lemma 1.*Let a dynamical system* $(\Re^3, S_t)$ *be asymptotically compact. Then for any bounded set* B *of* $\Re^3$ *the* $\omega-$*limit set* $\omega(B)$ *is a nonempty compact invariant set.*

Theorem 1.*Assume that a dynamical system* $(\Re^3, S_t)$ *is dissipative and asymptotically compact.Let* B *be a bounded absorbing set of the system* $(\Re^3, S_t)$*.Then the set* $A = \omega(B)$ *is a nonempty compact set and is a global attractor of the dynamical system* $(\Re^3, S_t)$*.The attractor* A *is a connected set in* $\Re^3$.

(for more details see [2] pag 140).

Noted that **Theorem 1** along with **Lemma 1** gives the folowing criterion: *a dynamical system possesses a compact global attractor if and only if it is asymptotically compact.* We introduce the polar coordinates:

$y(t) = r(t) \cdot \cos(\phi(t)); z(t) = r(t) \cdot \sin(\phi(t)); \phi(t) = -\beta \cdot t + \phi_0$.Hence , the system (1.2) can be rewritten in the following form:

$$\begin{cases} x' + \alpha \cdot x + r^2 = 0 \\ r' + \gamma \cdot r - x \cdot r = 0 \end{cases} ; \alpha, \beta \in \Re \tag{1.3}$$

which has a stationary point in $\{x = 0, r = 0\}$ for all $\alpha > 0$ and $\gamma \in \mathbb{R}$. If $\gamma < 0$ then one more stationary point $\{x = \gamma, r = \sqrt{-\alpha \cdot \gamma}\}$occurs in system (1.3).It corresponds to a periodic trajectory of the original problem (1.1).

We show using Poincare's method that the point (0,0) is a stable node of system (1.3) when $\gamma > 0$ and it is saddle when $\gamma < 0$.We compute:

$$A = J(F)_{(0,0)} = \begin{pmatrix} \alpha & 2 \cdot r \\ -r & \gamma - x \end{pmatrix}_{(0,0)} = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}$$

Hence $\det(A) = \alpha \cdot \gamma; trace(A) = \alpha + \gamma$; Obviously $(trace(A))^2 > 4 \cdot \det(A)$.

We observe that if $\gamma > 0$ then $\det(A) > 0$ and (0,0) is stable node,otherwise if $\gamma < 0$ then $\det(A) < 0$ and (0,0) is saddle point.

In the same way we study the nature of stationary point $\{x = \gamma,\, r = \sqrt{-\alpha \cdot \gamma}\}$.We compute :

$$A = J(F)_{(\gamma, \sqrt{-\alpha \cdot \gamma})} = \begin{pmatrix} \alpha & 2 \cdot \sqrt{-\alpha \cdot \gamma} \\ -\sqrt{-\alpha \cdot \gamma} & 0 \end{pmatrix}$$

hence, $\det(A) = -2 \cdot \alpha \cdot \gamma; trace(A) = \alpha$.Obviously $(trace(A))^2 > 4 \cdot \det(A)$ if only if $\alpha + 8 \cdot \gamma > 0$.

Further more if $\gamma < 0$ then $\det(A) > 0$ and $(\gamma, \sqrt{-\alpha \cdot \gamma})$ is stable nod ,otherwise if $\gamma > 0$ $(\gamma, \sqrt{-\alpha \cdot \gamma})$ is saddle point.

If $\gamma < -\alpha/8$ and $\alpha \neq 0$ then $(\gamma, \sqrt{-\alpha \cdot \gamma})$ is focus (for $\alpha = 0$ is centre).

A) If $\gamma > 0$ then (1.3) implies that

$$\tfrac{1}{2} \cdot \tfrac{d}{dt} \cdot (x^2 + r^2) + \min(\alpha, \gamma)(x^2 + r^2) \leq 0$$

Using again **Gronwall's lema** we obtain:

$$x(t)^2 + r(t)^2 \leq (x(0)^2 + r(0)^2) \exp(-2 \min(\alpha, \gamma) \cdot t)$$

Due $\min(\alpha, \gamma) > 0$ implies $\exp(-2 \min(\alpha, \gamma) \cdot t) \to \infty$ if $t \to \infty$ then

$x(t)^2 + r(t)^2 \leq 0$ hence, the global attractor of the system $(\Re^3, S_t)$ consists of the single stationary exponentially attracting point $(0, 0, 0)$.

B)If $\gamma = 0$ then $\min(\alpha, \gamma) = 0$ hence,

$$x(t)^2 + r(t)^2 \leq (x(0)^2 + r(0)^2), \det(A) = 0; (trace(A))^2 = \alpha^2 > 0$$

then global attractor of Cauchy problem (1.1) consists of the single stationary point $(0, 0, 0)$ but it is not exponentially attracting because it is a saddle point.

C)Now we consider the case $\gamma < 0$.Let us again refer to problem (1.3).It is clear that the line $r = 0$ is a stable manifold of the stationary

$\{x = 0, r = 0\}$Moreover ,it is obvious that if $r(t_0) > 0$,then the value $r(t)$ remains positive for all $t > t_{0.}$ Therefore ,the Lyapunov's function :

$$V(x, r) = \tfrac{1}{2}(x - \gamma)^2 + \tfrac{1}{2} \cdot r^2 + \gamma \cdot \alpha \cdot \ln r \qquad (1.4)$$

is defined on all the trajectories, the initial point of which does not lie on the line $r = 0$.Simple calculation show that :

$$\tfrac{d}{dt}(V(x(t), r(t)) + \alpha(x(t) - \gamma)^2 = 0 \qquad (1.5)$$

and

$$V(x, r) \geq V(\gamma, \sqrt{-\alpha \cdot \gamma}) + \tfrac{1}{2}((x - \gamma)^2 + (r - \sqrt{-\alpha \cdot \gamma})^2) \qquad (1.6)$$

therewith $V(\gamma, \sqrt{-\alpha \cdot \gamma}) = (1/2)\alpha \cdot |\gamma| \cdot \ln(e/(\alpha \cdot |\gamma|))$.Equation (1.5) implies that the function $V(x, r)$ does not increase along the trajectories. Therefore, any semitrajectory$\{(x(t); r(t)), t \in \Re_+\}$ starting from the point $\{x_0, r_0; r_0 \neq 0\}$ of the system $(\Re^2 \times \Re_+, S_t)$ generated by equation (1.3) possesses the property $V(x(t), r(t)) \leq V(x_0, r_0)$ for $t \geq 0$. Therewith, equation (1.4) implies that this semitrajectory can not approach the line $r = 0$ at the distance less then

$$\exp\{[1/(\alpha \cdot \gamma)] \cdot V(x_0, r_0)\}.$$

Hence ,this semitrajectories tends to $\bar{y} = \{x = \gamma, r = \sqrt{-\alpha \cdot \gamma}\}$. Moreover, for any $\xi \in \Re$ the set:

$$B\xi = \{y = (x, r) : V(x, r) \leq \xi\}$$

is uniformly attracted to $\bar{y}$, i.e. for any $\varepsilon > 0$ there exists $t_0 = t_0(\xi, \varepsilon)$ such that

$$S_t B\xi \subset \{y : \left|y - \bar{y}\right| \leq \varepsilon\}.$$

Indeed ,if it is not true,then there exists $\varepsilon_0 > 0$, a sequence $t_n \to \infty$, and $z_n \in B\varepsilon$ such that $\left|S_{t_n} \cdot z_n - \bar{y}\right| > \varepsilon_0$. The monotonicity of $V(y)$ and property (1.6) implies that

$$V(S_t \cdot z) \geq V(S_{t_n} \cdot z_n) \geq V(\gamma, \sqrt{-\alpha \cdot \gamma}) + \tfrac{1}{2}\varepsilon_0^2$$

for all $0 \leq t \leq t_n$. Let z be a limit point of sequence $\{z_n\}$. Then after passing to the limit we find out that:

$$V(S_t \cdot z) \geq V(\gamma, \sqrt{-\alpha \cdot \gamma}) + \tfrac{1}{2}\varepsilon_0^2; t \geq 0$$

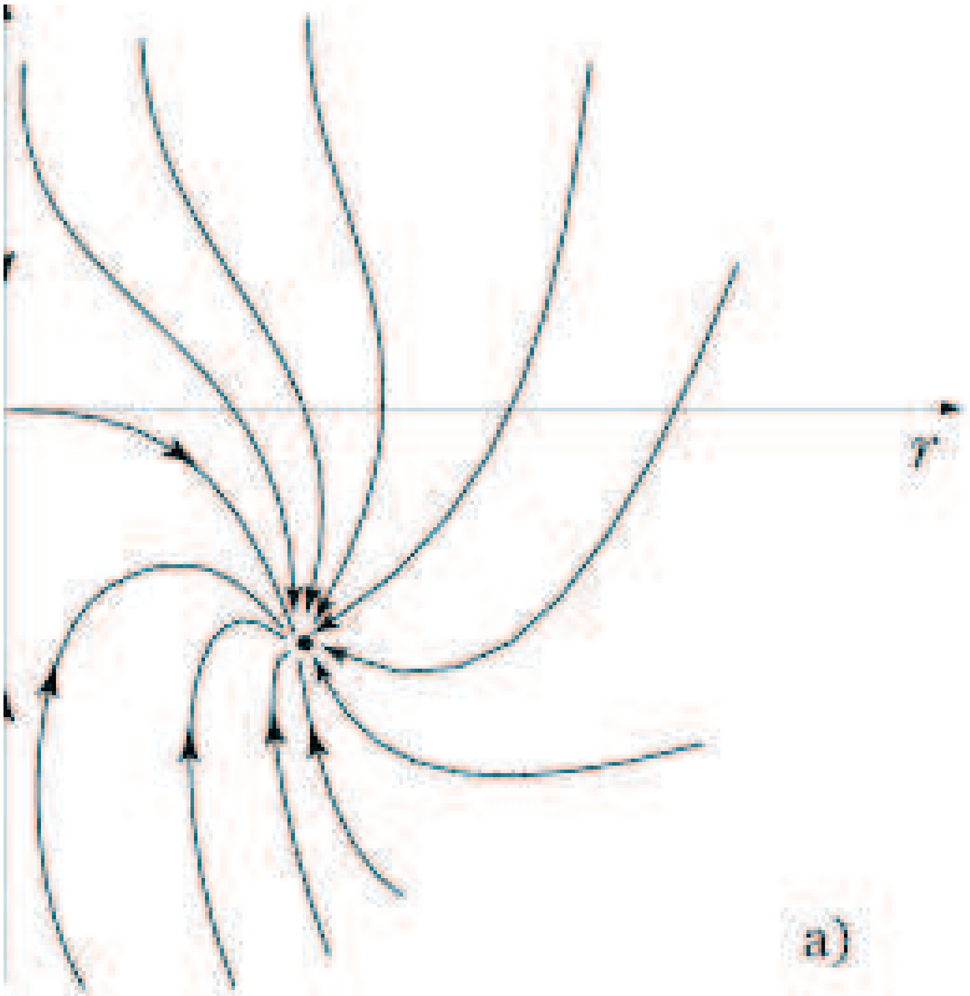
with $z \notin \{r = 0\}$.Thus ,the least inequality is impossible since

$$S_t \cdot z \to \bar{y} = \{x = \gamma, r = \sqrt{-\alpha \cdot \gamma}\}$$

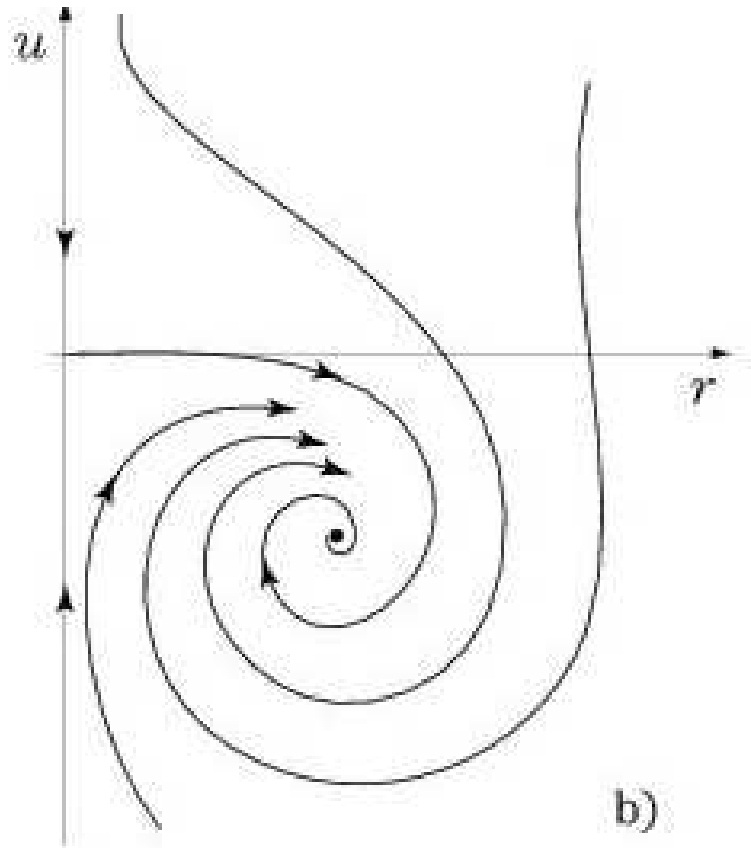
3 The qualitative behavior of solutions

For showing the qualitative behavior of solutions of Cauchy's problem (1.3) on the semiplan,we generate a script using Matlab 7.0.

In the figure 1 we represent the trajectories in cases: a) $\alpha = 10$; $\gamma = -0.0625$ stable nod; b)$\alpha = 10; \gamma = -6.25$ focus.



a)



References

[1] Babin A. V., Vishick M. I., *Attractors of evolution equations*, Amsterdam, North Holland 1992.

[2] Eden A., Foias C., Nicolaenko B.,Temam R., *Exponential attractors for dissipative evolution equation*, New York, Willey, 1994.

[3] Hale J.K. *Assimptotic behaviour of dissipative systems*, American Math. Soc., Vol 10, pag. 75-80, 1988.

[4] Hairer E., Norset S.P., Wanner G., *Solving ordinary differential equations*, Springer Verlag Berlin, Heidelberg, New-York, 2000, pag 80-104.

[5] Stuart A.M., Humphries A.R., *Dynamical Systems and Numerical Analysis*, Cambridge University Press 1996, pag. 81-87, 180-191, 398-411, 510-520.

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