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On the Convergence of implicit Ishikawa Iterations with Errors to a Common Fixed Point of Two Mappings in Convex Metric Spaces [1]

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Abstract

Let C be a convex subset of a complete generalized convex metric space X, and S and T be two self mappings on C. In this paper it is shown that if the sequence of modified Ishikawa iterations with errors in the sense of Xu [19] associated with S and T converges, then its limit point is the common fixed point of S and T. This result extends and generalizes the corresponding results of Niampally and Singh [10], Rhoades [12], Hicks and Kubicek [6] and Ciric et al [3].

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1 Introduction

Takahashi [18] introduced a notion of convex metric spaces and studied the fixed-point theory for nonexpansive mappings in such setting for the convex metric spaces, Kirk [8] and Goebel and Kirk [5] used the term "hyperbolic type space" when they studied the iteration processes for nonexpansive mappings in the abstract framework. For the Banach space, Petryshyn and Williamson [11], in 1973 proved a sufficient and necessary condition for Picard iterative sequences and Mann iterative sequences to converge to fixed points for quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [4] extended the results of [11] and gave the sufficient and necessary condition for Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings.

In recent years several authors have studied the convergence of the sequence of the Mann iterates [9] of a mapping T to a fixed point of T, under various contractive conditions.

The Ishikawa iteration scheme [7] was first used to establish the strong convergence for a pseudocontractive selfmapping of a convex compact subset of a Hilbert space. Very soon both iterative processes were used to establish the strong convergence of the respective iterates for some contractive type mappings in Hilbert spaces and then in more general normed linear spaces.

Naimpally and Singh [10] have studied the mappings which satisfy the contractive definition introduced in [2]. They proved the following theorem.

Theorem 1.1. *Let X be a normed linear space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be a selfmapping satisfying*

$$(NS) \qquad \|Tx - Ty\| \leq$$

$$\leq h \max \{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\}$$

for all x, y in C, where $0 \leq h < 1$ and let $\{x_n\}$ be the sequence of the Ishikawa scheme associated with T, that is $x_0 \in C$,

$$y_n = (1-\beta_n)x_n + \beta_n T x_n,$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \ n \geq 0, \tag{1}$$

where $0 \leq \alpha_n, \beta_n \leq 1$. If $\{\alpha_n\}$ is bounded away from zero and if $\{x_n\}$ converges to p, then p is a fixed point of T.

Definition 1.1. *[18.] Let (X,d) be a metric space. A mapping $W : X \times X \times [0,1] \to X$ is said to be a convex structure on X if for each $(x,y,\lambda) \in X \times X \times [0,1]$ and $u \in X$,*

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y). \tag{T}$$

The metric space X together with W is called a convex metric space.

Definition 1.2. *Let X be a convex metric space. A nonempty subset A of X is said to be convex if $W(x,y,\lambda) \in A$ whenever $(x,y,\lambda) \in A \times A \times [0,1]$.*

Remark 1.1. *(Takahashi [18]) has shown that open spheres and closed spheres are convex.*

Remark 1.2. *All normed spaces and their convex subsets are convex metric spaces. But there are many examples of convex metric spaces which are not embedded in any normed space(see Takahashi [18]).*

Remark 1.3. *Clearly a Banach space , or any convex subset of it, is a convex metric space with*

$$W(x,y,\lambda) = \lambda x + (1-\lambda)y.$$

More generally, if X is a linear space with a translation invariant metric satisfying $d(\lambda x+(1-\lambda)y,0) \leq \lambda d(x,0)+(1-\lambda)d(y,0)$, then X is a convex metric space.

Remark 1.4. *It is clear from (T) that*

$$\begin{aligned} d[x,W(x,y,\lambda)] &= (1-\lambda)d(x,y), \\ d[y,W(x,y,\lambda)] &= \lambda d(x,y). \end{aligned}$$

Recently Ciric et al [3] generalized the results of Naimpally and Singh to a pair of mappings S and T, defined on a convex metric space. They proved that if the sequence of Ishikawa iterations associated with S and T converges, then its limit point is the common fixed point of S and T, which satisfy the following condition:

(CUK) $$d(Sx,Ty) \leq h[d(x,y)+d(x,Ty)+d(y,Sx)],$$

where $0<h<1$. It is clear that the condition (CUK) is very general, since by the triangle inequality, condition (CUK) is always satisfied with $h=1$.

Inspired and motivated by the above said facts, we are introducing the following new concepts.

Definition 1.3. *Let (X,d) be a metric space. A mapping $W: X\times X\times X\times[0,1]\times[0,1]\times[0,1]\to X$ is said to be a generalized convex structure on X if for each $(x,y,z;a,b,c)\in X\times X\times X\times[0,1]\times[0,1]\times[0,1]$ and $u\in X$,*

(2) $$d(u,W(x,y,z;a,b,c)) \leq ad(u,x)+bd(u,y)+cd(u,z);$$

$a+b+c=1$. The metric space X together with W is called a generalized convex metric space.

Definition 1.4. *Let X be a generalized convex metric space. A nonempty subset A of X is said to be generalized convex if $W(x, y, z; a, b, c) \in A$ whenever $(x, y, z; a, b, c) \in A \times A \times A \times [0, 1] \times [0, 1] \times [0, 1]$.*

Remark 1.5. *Clearly every generalized convex metric space is a convex metric space.*

Remark 1.6. *Clearly every generalized convex set is a convex set.*

Remark 1.7. *It can be easily seen that open spheres and closed spheres are generalized convex.*

Remark 1.8. *All normed spaces and their generalized convex subsets are generalized convex metric spaces.*

Remark 1.9. *Clearly a Banach space , or any generalized convex subset of it, is a generalized convex metric space with $W(x, y, z; a, b, c) = ax + by + cz$. More generally, if X is a linear space with a translation invariant metric satisfying $d(ax + by + cz, 0) \leq ad(x, 0) + bd(y, 0) + cd(z, 0)$, then X is a generalized convex metric space.*

Remark 1.10. *It is clear from (1.2) that*

$$\begin{aligned} d[x, W(x, y, z; a, b, c)] &\leq bd(x, y) + cd(x, z), \\ d[y, W(x, y, z; a, b, c)] &\leq ad(x, y) + cd(y, z), \\ d[z, W(x, y, z; a, b, c)] &\leq ad(x, z) + bd(y, z). \end{aligned} \tag{3}$$

Let C be a nonempty closed convex subset of a generalized convex metric space X.

The sequence $\{x_n\}$ defined by

$$x_0 \in C, x_n = W(x_{n-1}, Sy_n, u_n; a_n, b_n, c_n) \tag{4}$$

$$y_n = W(x_{n-1}, Tx_n, v_n; a'_n, b'_n, c'_n), \;\; n \geq 1,$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C, is called Xu-Ishikawa [19] type iteration process.

Remark 1.11. *Clearly, the iteration process (1.4) contains all the known iteration processes [7, 9, 19] as its special case.*

The sequence $\{x_n\}$ defined by

$$x_0 \in C, x_n = W(x_{n-1}, STx_n, u_n; a_n, b_n, c_n), \;\; n \geq 1, \tag{5}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = 1$ and $\{u_n\}$ is a bounded sequence in C, is called Xu-Mann [19] type iteration process.

The sequence $\{x_n\}$ defined by

$$x_0 \in C, x_n = W(x_{n-1}, Sy_n, \alpha_n), y_n = W(x_{n-1}, Tx_n, \beta_n), \;\; n \geq 1, \tag{6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$, is called implicit Ishikawa type iteration process.

The sequence $\{x_n\}$ defined by

$$x_0 \in C, x_n = W(x_{n-1}, STx_n, \alpha_n), \;\; n \geq 1, \tag{7}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$, is called implicit Mann type iteration process.

The purpose of this paper is to generalize the results of Naimpally and Singh and Ciric et al to a pair of mappings S and T, defined on a generalized convex metric space. It is shown that if the sequence (4) converges, then its limit point is the common fixed point of S and T, which satisfy the condition (CUK).

2 Main Results

Now we prove our main results.

Theorem 2.1. *Let C be a nonempty closed convex subset of a generalized convex metric space X and let S, $T : X \to X$ be selfmappings satisfying (CUK) for all x, y in C. Suppose that $\{x_n\}$ is defined by (4) satisfying $\{b_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$ and $\lim_{n\to\infty} c_n = 0 = \lim_{n\to\infty} c_n'$. If $\{x_n\}$ converges to some point $p \in C$, then p is the common fixed point of S and T.*

Let
$$M = \max\left\{\sup_{n\geq 1} d(p,u_n), \sup_{n\geq 1} d(p,v_n)\right\}.$$

From (3-4) and using triangle inequality, we have

$$\begin{aligned}
d(x_{n-1}, Sy_n) &\leq d(x_{n-1}, x_n) + d(x_n, Sy_n) \\
&= d(x_{n-1}, x_n) + d(W(x_{n-1}, Sy_n, u_n; a_n, b_n, c_n), Sy_n) \\
&\leq d(x_{n-1}, x_n) + a_n d(x_{n-1}, Sy_n) + c_n d(Sy_n, u_n) \\
&\leq d(x_{n-1}, x_n) + (a_n + c_n) d(x_{n-1}, Sy_n) + c_n d(x_{n-1}, u_n) \\
&= d(x_{n-1}, x_n) + (1-b_n) d(x_{n-1}, Sy_n) + c_n d(x_{n-1}, u_n),
\end{aligned}$$

implies

$$d(x_{n-1}, Sy_n) \leq \frac{1}{b_n} d(x_{n-1}, x_n) + \frac{c_n}{b_n} d(x_{n-1}, u_n).$$

With the help of condition $\{b_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$, we get

$$d(x_{n-1}, Sy_n) \leq \frac{1}{\delta} d(x_{n-1}, x_n) + \frac{c_n}{\delta} d(x_{n-1}, u_n).$$

Since $x_n \to p$, $d(x_{n-1}, x_n) \to 0$ and $\lim_{n\to\infty} c_n = 0$, implies

$$\lim_{n\to\infty} d(x_{n-1}, Sy_n) = 0. \tag{8}$$

Using (CUK) we get

$$d(Sy_n, Tx_n) \leq h[d(y_n, x_n) + d(y_n, Tx_n) + d(x_n, Sy_n)]. \tag{9}$$

Using triangle inequality, we have

$$\begin{aligned} d(y_n, x_n) &\leq d(y_n, x_{n-1}) + d(x_{n-1}, x_n) = \\ &= d(W(x_{n-1}, Tx_n, v_n; a'_n, b'_n, c'_n), x_{n-1}) + d(x_{n-1}, x_n) \leq \\ &\leq b'_n d(x_{n-1}, Tx_n) + c'_n d(x_{n-1}, v_n) + d(x_{n-1}, x_n). \end{aligned} \tag{10}$$

Also

$$\begin{aligned} d(y_n, Tx_n) &= d(W(x_{n-1}, Tx_n, v_n; a'_n, b'_n, c'_n), Tx_n) \leq \\ &\leq a'_n d(x_{n-1}, Tx_n) + c'_n d(Tx_n, v_n). \end{aligned} \tag{11}$$

Substituting (10-11) in (9), we get

$$\begin{aligned} d(Sy_n, Tx_n) &\leq h[d(x_{n-1}, x_n) + d(x_n, Sy_n) + (a'_n + b'_n) d(x_{n-1}, Tx_n) \\ &\quad + c'_n d(x_{n-1}, v_n) + c'_n d(Tx_n, v_n)] \\ &\leq h[d(x_{n-1}, x_n) + d(x_n, Sy_n) + (1 - c'_n) d(x_{n-1}, Tx_n) \\ &\quad + c'_n d(x_{n-1}, v_n) + c'_n d(Tx_n, x_{n-1}) + c'_n d(x_{n-1}, v_n)] \\ &= h[d(x_{n-1}, x_n) + d(x_n, Sy_n) + d(x_{n-1}, Tx_n) + 2c'_n d(x_{n-1}, v_n)] \\ &\leq h[d(x_{n-1}, x_n) + d(x_n, Sy_n) + d(x_{n-1}, Sy_n) + d(Sy_n, Tx_n) \\ &\quad + 2c'_n d(x_{n-1}, v_n)], \end{aligned}$$

implies

$$\begin{aligned} d(Sy_n, Tx_n) &\leq \frac{h}{1-h}\left[d(x_{n-1}, x_n) + d(x_n, Sy_n) + d(x_{n-1}, Sy_n) + \right. \\ &\left. +2c_n^{'} d(x_{n-1}, v_n)\right]. \end{aligned}$$

Taking the limit as $n \to \infty$ we obtain, by $x_n \to p$, $Sy_n \to p$ and $\lim_{n\to\infty} c_n^{'} = 0$,

$$\lim_{n\to\infty} d(Sy_n, Tx_n) = 0.$$

Since $Sy_n \to p$, it follows that $Tx_n \to p$.

Since

$$\begin{aligned} d(y_n, x_{n-1}) &= d(W(x_{n-1}, Tx_n, v_n; a_n', b_n', c_n'), x_{n-1}) \\ &\leq b_n^{'} d(x_{n-1}, Tx_n) + c_n^{'} d(x_{n-1}, v_n). \end{aligned}$$

Taking the limit as $n \to \infty$, it follows also that $y_n \to p$.

From (CUK) again, we have

$$d(Sy_n, Tp) \leq h[d(y_n, p) + d(y_n, Tp) + d(p, Sy_n)].$$

Taking the limit as $n \to \infty$ we obtain $d(p, Tp) \leq hd(p, Tp)$.

Since $h < 1$, $d(p, Tp) = 0$. Hence $Tp = p$. Similarly, from (CUK),

$$d(Sp, Tx_n) \leq h[d(p, x_n) + d(p, Tx_n) + d(x_n, Sp)].$$

Taking the limit as $n \to \infty$ we get $d(Sp, p) \leq hd(p, Sp)$.

Hence $Sp = p$. Therefore $Sp = Tp = p$. This completes the proof.

Theorem 2.2. *Let C be a nonempty closed convex subset of a generalized convex metric space X and let S, $T : X \to X$ be selfmappings satisfying (CUK) for all x, y in C. Suppose that $\{x_n\}$ is defined by (5) satisfying $\{b_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$ and $\lim_{n\to\infty} c_n = 0$. If $\{x_n\}$ converges to some point $p \in C$, then p is the common fixed point of S and T.*

Theorem 2.3. *Let X be a normed linear space and C be a closed convex subset of X and let $S, T : X \to X$ be selfmappings satisfying (CUK) for all x, y in C. Suppose that $\{x_n\}$ is defined by*

$$x_0 \in C, x_n = a_n x_{n-1} + b_n S y_n + c_n u_n, y_n = a'_n x_{n-1} + b'_n T x_n + c'_n v_n, \ n \geq 1,$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ are sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$, satisfying $\{b_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$, $\lim\limits_{n\to\infty} c_n = 0 = \lim\limits_{n\to\infty} c'_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. If $\{x_n\}$ converges to some point $p \in C$, then p is the common fixed point of S and T.

Theorem 2.4. *Let X be a normed linear space and C be a closed convex subset of X and let $S, T : X \to X$ be selfmappings satisfying (CUK) for all x, y in C. Suppose that $\{x_n\}$ is defined by*

$$x_0 \in C, x_n = a_n x_{n-1} + b_n ST x_n + c_n u_n, \ n \geq 1,$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1$, satisfying $\{b_n\} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$, $\lim\limits_{n\to\infty} c_n = 0$ and $\{u_n\}$ is a bounded sequence in C. If $\{x_n\}$ converges to some point $p \in C$, then p is the common fixed point of S and T.

Corollary 2.1. *Let X be a normed linear space and C be a closed convex subset of X. Let $S, T : C \to C$ be two mappings satisfying (CUK) and $\{x_n\}$ be the sequence of Ishikawa scheme associated with S and T; for $x_0 \in C$,*

$$\begin{aligned} y_n &= \beta_n x_{n-1} + (1 - \beta_n) T x_n, \\ x_n &= \alpha_n x_{n-1} + (1 - \alpha_n) S y_n, \ n \geq 1. \end{aligned}$$

If $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$ and $\{x_n\}$ converges to p, then p is a common fixed point of S and T.

Corollary 2.2. *Let X be a normed linear space and C be a closed convex subset of X. Let $S, T : C \to C$ be two mappings satisfying (CUK) and $\{x_n\}$ be the sequence of Ishikawa scheme associated with S and T; for $x_0 \in C$,*

$$\begin{aligned} y_n &= \beta_n x_{n-1} + (1-\beta_n) T x_n, \\ x_n &= \alpha_n x_{n-1} + (1-\alpha_n) S y_n,\ n \geq 1. \end{aligned}$$

If $\lim_{n\to\infty} \alpha_n = 0$ and $\{x_n\}$ converges to p, then p is a common fixed point of S and T.

Corollary 2.3. *Let X be a normed linear space and C be a closed convex subset of X. Let $S, T : C \to C$ be two mappings satisfying (CUK) and $\{x_n\}$ be the sequence of Mann scheme associated with S and T; for $x_0 \in C$,*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) ST x_n,\ n \geq 1.$$

If $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$ and $\{x_n\}$ converges to p, then p is a common fixed point of S and T.

Corollary 2.4. *Let X be a normed linear space and C be a closed convex subset of X. Let $S, T : C \to C$ be two mappings satisfying (CUK) and $\{x_n\}$ be the sequence of Mann scheme associated with S and T; for $x_0 \in C$,*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) ST x_n,\ n \geq 1.$$

If $\lim_{n\to\infty} \alpha_n = 0$ and $\{x_n\}$ converges to p, then p is a common fixed point of S and T.

Remark 2.1. *Corollaries with $S = T$ are the generalizations of Theorem 1 of Niampally and Singh [10]. Therefore, Theorem 9 of Rhoades [14] is also a special case of corollary where (CUK) is replaced with the following condition, introduced in [1],*

$$d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

where $0 < h < 1$.

References

[1] Ciric Lj. B., *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. 45 (1974), 267-273.

[2] Ciric Lj. B., *Quasi-contractions in Banach spaces*, Publ. Inst. Math. 21 (1977), 41-48.

[3] Ciric Lj. B et al, *On the convergence of the ishikawa iterates to a common fixed point of two mappings*, Archivum Mathematicum (Brno), Tomus 39 (2003), 123-127.

[4] Ghosh M. K., Debnath L., *Convergence of Ishikawa iterates of quasl-nonexpansive mappings*, J. Math. Anal. Appl. 207, 96-103, (1997).

[5] Goebel K., Kirk W. A., *Iteration processes for nonexpansive mappings*, In Contemporary Math, Volume 21, pp. 115-123, Amer Math. Soc, Providence, RI, (1983).

[6] Hichs L., Kubicek, J. D., *On the Mann iteration process in Hilbert spaces*, J. Math. Anal. Appl. 59 (1977), 498-504.

[7] Ishikawa S.,*Fixed points by a new iteration method*, Proc. Amer. Math. Soc. 44 (1974), 147-150.

[8] Kirk W. A., *Krasnoselskfi's iteration process in hyperbohc space*, Number, Funet., Anal. Optim 4, 371-381, (1982).

[9] Mann W. R., *Mean value methods in iteration*, Proc. Amer. Math. Soc. 4 (1953), 506-510.

[10] Naimpally S. A., Singh K. L., *Extensions of some fixed point theorems of Rhoades*, J. Math. Anal. Appl. 96 (1983), 437-446.

[11] Petryshyn W. V., Williamson T. E., *Strong and weak convergence of the sequence of the sequence of successive apprommations for quasl-nonexpanslve mappmgs*, J. Math. Anal. Appl. 43, 459-497, (1973).

[12] Rhoades B. E., *Fixed point iterations using infinite matrices*, Trans. Amer. Math. Soc. 196 (1974), 161-176.

[13] Rhoades B. E., *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. 226 (1977), 257-290.

[14] Rhoades B. E., *Extension of some fixed point theorems of Ciric, Maiti and Pal*, Math. Sem. Notes Kobe Univ. 6 (1978), 41-46.

[15] Rhoades B. E., *Comments on two fixed point iteration methods*, J. Math. Anal. Appl. 56 (1976), 741-750.

[16] Singh K. L., *Fixed point iteration using infinite matrices*, In "Applied Nonlinear Analysis", (V. Lakshmikantham, Ed.), pp.689{703, Academic Press, New York, 1979.

[17] Singh K. L., *Generalized contractions and the sequence of iterates*, In "Nonlinear Equations in Abstract Spaces" (V. Lakshmikantham, Ed.), pp. 439-462, Academic Press, New York, 1978.

[18] Takahashi W., *A convexity in metric spaces and nonexpansive mappings*, Kodai Math. Sem. Rep. 22 (1970), 142-149.

[19] Xu Y., *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl. **224** (1998), 91-101.

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