General Mathematics Vol. 14, No. 3 (2006), 43–52

Angular estimates of analytic functions defined by Carlson - Shaffer linear operator [1]

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Abstract

The object of the present paper is to derive some argument properties of analytic functions defined by the Carlson - Shaffer linear operator $L(a, c)f(z)$. Our results contain some interesting corollaries as the special cases.

2000 Mathematics Subject Classification: 30C45.

Keywords: Analytic functions, Argument, Hadamard product, linear operator.

[1] *Received 12 July, 2006*

Accepted for publication (in revised form) 15 September, 2006

1 Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. For two functions $f(z)$ and $g(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.2}$$

their Hadamard product (or convolution) is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.3}$$

Define the function $\phi(a, c; z)$ by

$$\phi(a, c; z) \quad : \quad = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \tag{1.4}$$

$$(a \in \mathbb{R};\ c \in \mathbb{R}\backslash\mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0, -1, -2, \ldots\},\ z \in \mathcal{U}),$$

where $(\lambda)_n$ is the Pochhammer symbol given, in terms of Gamma functions,

$$(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0, \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1), & n \in \mathbb{N} : \{1, 2, \ldots\}. \end{cases}$$

Corresponding to the function $\phi(a, c; z)$, Carlson and Shaffer[1] introduced a linear operator $L(a, c) : \mathcal{A} \to \mathcal{A}$ by

$$L(a, c)f(z) := \phi(a, c; z) * f(z), \tag{1.5}$$

or, equivalently, by

$$L(a,c)f(z) := z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1} \qquad (z \in \mathcal{U}). \tag{1.6}$$

It follows from (1.6) that

$$z(L(a,c)f(z))' = aL(a+1,c)f(z) - (a-1)L(a,c)f(z), \tag{1.7}$$

and $L(1,1)f(z) = f(z)$, $L(2,1)f(z) = zf'(z)$, $L(3,1)f(z) = zf'(z) + \frac{1}{2}z^2 f''(z)$.

Many properties of analytic functions defined by the Carlson-Shaffer linear operator were studied by (among others) Owa and Srivastava [7], Ding [3], Kim and Lee [4], Ravichandran *et al.*[6] and Shanmugam *et al.* [5].

In this paper we shall derive some argument properties of analytic functions defined by the linear operator $L(a,c)f(z)$.

In order to prove our main results, we recall the following lemma:

Lemma 1.1.([2]). Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ $(z \in \mathcal{U})$ and suppose that

$$|\arg(p(z) + \beta z p'(z))| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\alpha\beta\right) \qquad (\alpha > 0,\ \beta > 0), \tag{1.8}$$

then we have

$$|\arg p(z)| < \frac{\pi}{2}\alpha \qquad (z \in \mathcal{U}). \tag{1.9}$$

2 Main Results

Theorem 2.1. Let $a+1>\mu>0$, $\alpha>0$, λ is any real number , $L(a,c)f(z)/L(a+1,c)f(z)\neq 0$ $(z\in\mathcal{U})$ and suppose that

$$\left|\arg\left(\frac{(a+1)L(a,c)f(z)}{(a+1-\mu)L(a+1,c)f(z)}\left[\lambda\frac{L(a+1,c)f(z)}{L(a,c)f(z)}-\mu\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}+1\right]\right)\right.$$

$$\left.-\left(\frac{(a+1)\lambda-a\mu}{a+1-\mu}\right)\right|<\frac{\pi}{2}\left(\alpha+\frac{2}{\pi}\tan^{-1}\frac{\mu}{a+1-\mu}\alpha\right) \tag{2.1}$$

then we have

$$\left|\arg\left(\frac{L(a,c)f(z)}{L(a+1,c)f(z)}\right)\right|<\frac{\pi}{2}\alpha \qquad (z\in\mathcal{U}). \tag{2.2}$$

Proof. Define the function $p(z)$ by

$$p(z):=\frac{L(a,c)f(z)}{L(a+1,c)f(z)}. \tag{2.3}$$

Then $p(z)=1+b_1z+b_2z+\cdots$ is analytic in $\mathcal{U}$ with $p(0)=1$ and $p(z)\neq 0$ $(z\in\mathcal{U})$. Also, by a simple computation, we find from (2.3) that

$$\frac{zp'(z)}{p(z)}=\left(\frac{z(L(a,c)f(z))'}{L(a,c)f(z)}-\frac{z(L(a+1,c)f(z))'}{L(a+1,c)f(z)}\right) \tag{2.4}$$

by making use of the familiar identity (1.7) in (2.4) , we get

$$\begin{aligned}
&\left[\lambda\frac{L(a+1,c)f(z)}{L(a,c)f(z)}-\mu\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}+1\right]\frac{L(a,c)f(z)}{L(a+1,c)f(z)}\\
=&\left[\frac{\lambda}{p(z)}-\frac{\mu}{a+1}\left(1+\frac{a}{p(z)}-\frac{zp'(z)}{p(z)}\right)+1\right]p(z)\\
=&\frac{1}{a+1}[(a+1)\lambda-a\mu+(a+1-\mu)p(z)+\mu zp'(z)]
\end{aligned}$$

or, equivalently,

$$\frac{(a+1)L(a,c)f(z)}{(a+1-\mu)L(a+1,c)f(z)}\left[\lambda\frac{L(a+1,c)f(z)}{L(a,c)f(z)}-\mu\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}+1\right]$$

$$-\left(\frac{(a+1)\lambda-a\mu}{a+1-\mu}\right)=p(z)+\frac{\mu}{a+1-\mu}zp'(z). \tag{2.5}$$

The result of Theorem 2.1 now follows by an application of Lemma1.1.

Letting $a=c=1$ in Theorem 2.1, we have

Corollary 2.2. Let $2>\mu>0$, $\alpha>0$, λ is any real number , $f(z)/zf'(z)\neq 0$ $(z\in\mathcal{U})$ and suppose that

$$\left|\arg\left(\tfrac{2f(z)}{(2-\mu)zf'(z)}\left[\lambda\tfrac{zf'(z)}{f(z)}-\tfrac{\mu zf''(z)}{2f'(z)}+1-\mu\right]-\left(\tfrac{2\lambda-\mu}{2-\mu}\right)\right)\right|$$

$$<\frac{\pi}{2}\left(\alpha+\frac{2}{\pi}\tan^{-1}\frac{\mu}{2-\mu}\alpha\right) \tag{2.6}$$

then we have

$$\left|\arg\left(\frac{f(z)}{zf'(z)}\right)\right|<\frac{\pi}{2}\alpha \qquad (z\in\mathcal{U}). \tag{2.7}$$

Theorem 2.3. Let $a\neq -1$, $\lambda\neq-\mu$, α, $\lambda>0$, $\delta(a+1)(\lambda+\mu)>0$, $L(a+1,c)f(z)/z\neq 0$ $(z\in\mathcal{U})$ and suppose that

$$\left|\arg\left(\frac{1}{\lambda+\mu}\left(\frac{L(a+1,c)f(z)}{z}\right)^{\delta}\left(\lambda\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}+\mu\right)\right)\right|$$

$$<\frac{\pi}{2}\left(\alpha+\frac{2}{\pi}\tan^{-1}\frac{\lambda}{\delta(a+1)(\lambda+\mu)}\alpha\right) \tag{2.8}$$

then we have

$$\left|\arg\left(\frac{L(a+1,c)f(z)}{z}\right)^{\delta}\right|<\frac{\pi}{2}\alpha \qquad (z\in\mathcal{U}). \tag{2.9}$$

Proof. Define the function $p(z)$ by

$$p(z) := \left(\frac{L(a+1,c)f(z)}{z}\right)\delta. \tag{2.10}$$

Then $p(z) = 1 + b_1 z + b_2 z + \cdots$ is analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ $(z \in \mathcal{U})$. Also, by a simple computation and by making use of the familiar identity (1.7) we find from (2.10) that

$$\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} = \frac{1}{\delta(a+1)}\frac{zp'(z)}{p(z)} + 1 \tag{2.11}$$

by using (2.10) and (2.11), we get

$$\frac{1}{\lambda+\mu}\left(\frac{L(a+1,c)f(z)}{z}\right)\delta\left(\lambda\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} + \mu\right)$$

$$= p(z) + \frac{\lambda}{\delta(a+1)(\lambda+\mu)}zp'(z) \tag{2.12}$$

Using Lemma 1.1, we obtain the required result.

Letting $a = c = 1$ in Theorem 2.3, we have

Corollary 2.4. Let $\lambda \neq -\mu$, α, $\lambda > 0$, $2\delta(\lambda+\mu) > 0$, $f'(z) \neq 0$ $(z \in \mathcal{U})$ and suppose that

$$\left|\arg\left((f'(z))\,\delta\left(1 + \frac{\lambda z f''(z)}{2(\lambda+\mu)f'(z)}\right)\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\frac{\lambda}{2\delta(\lambda+\mu)}\alpha\right) \tag{2.13}$$

then we have

$$|\arg\,(f'(z))\,\delta| < \frac{\pi}{2}\alpha \qquad (z \in \mathcal{U}). \tag{2.14}$$

Theorem 2.5. Let $a \neq -1$, $\alpha, \lambda, \eta, \gamma > 0$, $L(a+1,c)f(z)/L(a,c)f(z) \neq 0$ $(z \in \mathcal{U})$ and suppose that

$$\arg\left|\left(\tfrac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)\gamma\left[\tfrac{\gamma\eta}{\lambda}\left((a+1)\tfrac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - a\tfrac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1\right) + 1\right]\right|$$

$$< \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\frac{\lambda}{\eta}\alpha\right) \tag{2.15}$$

then we have

$$\left|\arg\left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)\gamma\right| < \frac{\pi}{2}\alpha \qquad (z \in \mathcal{U}). \tag{2.16}$$

Proof. Define the function $p(z)$ by

$$p(z) := \left(\frac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)\gamma. \tag{2.17}$$

Then $p(z) = 1 + b_1 z + b_2 z + \cdots$ is analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ $(z \in \mathcal{U})$. Also, by a simple computation and by making use of the familiar identity (1.7), we find from (2.17) that

$$\left(\tfrac{L(a+1,c)f(z)}{L(a,c)f(z)}\right)\gamma\left[\tfrac{\gamma\lambda}{\eta}\left((a+1)\tfrac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - a\tfrac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1\right) + 1\right]$$

$$= p(z) + \frac{\lambda}{\eta}zp'(z) \tag{2.18}$$

An application of Lemma 1.1, we obtain the required result.

Letting $a = c = 1$ in Theorem 2.5, we have

Corollary 2.6. Let $\alpha, \lambda, \eta, \gamma > 0$, $zf'(z)/\ f(z) \neq 0$ $(z \in \mathcal{U})$ and suppose that

$$\left|\arg\left\{\left(\frac{zf'(z)}{f(z)}\right)\gamma\left[\frac{\gamma\eta}{\lambda}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) + 1\right]\right\}\right|$$

$$< \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\frac{\lambda}{\eta}\alpha\right) \tag{2.19}$$

then we have

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\gamma\right| < \frac{\pi}{2}\alpha \qquad (z \in \mathcal{U}). \tag{2.20}$$

Letting $\lambda = \eta = \gamma = 1$ in Corollary 2.6, we have

Corollary 2.7. Let $0 < \alpha \leq 1$, $zf'(z)/\ f(z) \neq 0\ \ (z \in \mathcal{U})$ and suppose that

$$\left|\arg\left\{\left(\frac{zf'(z)}{f(z)}\right)\left(2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right\}\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\alpha\right) \tag{2.21}$$

then we have

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\alpha \qquad (z \in \mathcal{U}), \tag{2.22}$$

that is, $f(z)$ is strongly starlike function of order α in $\mathcal{U}$.

Theorem 2.8. Let $a \neq -1$, $\alpha, \lambda, \gamma > 0$, $(\gamma - \lambda)(a + 1) > 0$, $z/L(a + 1, c)f(z) \neq 0$ $(z \in \mathcal{U})$ and suppose that

$$\begin{aligned} &\left|\arg\left[\frac{1}{\gamma - \lambda}\left(\gamma\frac{z}{L(a+1,c)f(z)} - \lambda z\frac{L(a+2,c)f(z)}{[L(a+1,c)f(z)]^2}\right)\right]\right| \\ &< \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\frac{\lambda}{(\gamma-\lambda)(a+1)}\alpha\right). \end{aligned} \tag{2.23}$$

then we have

$$\left|\arg\left(\frac{z}{L(a+1,c)f(z)}\right)\right| < \frac{\pi}{2}\alpha \qquad (z \in \mathcal{U}). \tag{2.24}$$

Proof. Define the function $p(z)$ by

$$p(z) := \frac{z}{L(a+1,c)f(z)} \tag{2.25}$$

Then $p(z) = 1 + b_1 z + b_2 z + \cdots$ is analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ $(z \in \mathcal{U})$. Also, by a simple computation and by making use of the familiar identity (1.7), we find from (2.25) that

$$\frac{1}{\gamma - \lambda}\left(\gamma \frac{z}{L(a+1,c)f(z)} - \lambda z \frac{L(a+2,c)f(z)}{[L(a+1,c)f(z)]^2}\right) = \tag{2.26}$$

$$= p(z) + \frac{\lambda}{(\gamma-\lambda)(a+1)} z p'(z).$$

The result of Theorem 2.8 now follows by an application of Lemma1.1.

Letting $a = c = 1$ in Theorem 2.8, we have

Corollary 2.9. Let $\alpha, \lambda, \gamma > 0$, $(\gamma - \lambda)(a+1) > 0$, $1/f'(z) \neq 0$ $(z \in \mathcal{U})$ and suppose that

$$\left|\arg\left(\frac{1}{f'(z)} - \frac{\lambda z f''(z)}{2(\gamma-\lambda)(f'(z))^2}\right)\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\frac{\lambda}{2(\gamma-\lambda)}\alpha\right) \tag{2.27}$$

then we have

$$\left|\arg\left(\frac{1}{f'(z)}\right)\right| < \frac{\pi}{2}\alpha \qquad (z \in \mathcal{U}). \tag{2.28}$$

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