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Weighted Ostrowski Type Inequality for Differentiable Mappings[1] whose first derivatives belong to $Lp(a, b)$

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Abstract

In this paper, we establish weighted Ostrowski type inequality for differentiable mappings whose first derivatives belong to $L_p(a, b)(p > 1)$. The inequality is also applied to numerical integration and for some special means.

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1 Introduction

Dragomir and Wang considered integral inequality of Ostrowski type for $\|.\|_p$ −norms $(p > 1)$ as follows [3]:

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\overset{0}{I}$ ($\overset{0}{I}$ is interior of I) and $a, b \in \overset{0}{I}$ with $a < b$. If $f' \in L_p(a,b)$ $\left(p > 1, \frac{1}{p} + \frac{1}{q} = 1\right)$, then, we have the inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq$$

$$\leq \frac{1}{b-a} \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{1/q} \|f'\|_p , \tag{1}$$

for all $x \in [a, b]$, *where*

$$\|f'\|_p = \left(\int_a^b |f'(t)|^p \, dt \right)^{1/p},$$

is the $\|.\|_p$ *−norm.*

They also pointed out some applications of (1.1) in numerical integration and for special means.

Ujević gave generalization of Ostrowski inequalities and applications in numerical integration in the form of the following theorem [7] .

Theorem 1.2. *Let $I \subset R$ be an open interval and $a, b \in I$, $a < b$. If $f : I \to R$ is a differentiable function such that $\gamma \leq f'(t) \leq \Gamma$, for all*

$t \in [a,b]$, for some constants $\gamma, \Gamma \in R$, then, we have

$$\left|\left[(1-\lambda)f(x)+\frac{f(a)+f(b)}{2}\lambda-\frac{\Gamma+\gamma}{2}(1-\lambda)(x-\frac{a+b}{2})\right](b-a)-\int_a^b f(t)dt\right| \leq$$

$$\leq \frac{\Gamma-\gamma}{2}\left\{\frac{(b-a)^2}{4}\left[\lambda^2+(1-\lambda)^2\right]+(x-\frac{a+b}{2})^2\right\}, \tag{2}$$

for all $\lambda \in [0,1]$ and $a+\lambda\frac{b-a}{2} \leq x \leq b-\lambda\frac{b-a}{2}$.

Fink [5] also obtained the following result for n-time differentiable functions.

Theorem 1.3. *Let $f^{n-1}(t)$ be absolutely continuous on [a,b] with $f^n(t) \in L_p(a,b)$ and let*

$$F_k(x) := \frac{n-k}{k!}\left[\frac{f^{k-1}(a)(x-a)^k - f^{k-1}(b)(x-b)^k}{b-a}\right],$$

where $k = 1, 2, \ldots, n-1$; then

$$\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}F_k(x)\right]-\frac{1}{b-a}\int_a^b f(y)dy \leq$$

$$\leq \begin{cases} \dfrac{\left[(x-a)^{nq+1}+(b-x)^{nq+1}\right]^{1/q}}{n!(b-a)}B^{1/q}(q+1,(n-1)q+1)\left\|f^n\right\|_p, \\ \text{for } f \in L_p[a,b],\ p>1,\ \dfrac{1}{q}+\dfrac{1}{p}=1, \\ \dfrac{(n-1)^{n-1}}{n^n n!(b-a)}\max\left\{(x-a)^n,(b-x)^n\right\}\left\|f^n\right\|_1, \\ \quad \text{for } f \in L_1[a,b], \\ \dfrac{(x-a)^{n+1}+(b-x)^{n+1}}{n(n-1)!(b-a)}\left\|f^n\right\|_\infty, \\ \text{for } f \in L_\infty[a,b], \end{cases} \tag{3}$$

where $B(\alpha,\beta)$ is Euler's Beta function,

$$\|f^n\|_p = \left[\int_a^b |f^n(t)|^p\right]^{1/p}, \text{ for } p \geq 1,$$

and

$$\| f^n\|_\infty = ess \sup_{t\in(a,b)} |f^n(t)|.$$

Dragomir and Sofo derived the generalization of the trapezoid formula for n-time differentiable functions [2].

Theorem 1.4. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that its $(n-1)th$ derivative f^{n-1} is absolutely continuous on $[a,b]$. Define*

$$T(a,b,n) :=$$

$$:= \frac{1}{n}\left[\frac{f(a)+f(b)}{2} + \sum_{k=1}^{n-1}\frac{(n-k)(b-a)^{k-1}}{k!}\left\{\frac{f^{k-1}(a)+(-1)^{k-1}f^{k-1}(b)}{2}\right\}\right] -$$

$$-\frac{1}{b-a}\int_a^b f(y)dy.$$

Then, we have

$$|T(a,b,n)| \leq \tag{4}$$

$$\leq \begin{cases} \dfrac{(x-a)^{n-1+1/q}}{n!} B^{1/q}(q+1,(n-1)q+1) \left\| f^n \right\|_p, \\ \text{for } f^n \in L_p[a,b],\ p,q>1,\ \dfrac{1}{q}+\dfrac{1}{p}=1, \\ \dfrac{(b-a)^{n-1/2}}{\sqrt{2(2n+1)!n!}} \left[2(2n-2)! + (-1)^n (n!)^2 \right]^{1/2} \left\| f^n \right\|_2, \\ \quad \text{for } f^n \in L_2[a,b],\ p=q=2, \\ \dfrac{(b-a)^n}{n(n+1)!} \left\| f^n \right\|_\infty, \\ \text{for } f^n \in L_\infty[a,b], \end{cases}$$

and for the interval $(a, \frac{a+b}{2}]$ *with even* n

$$|T(a,b,2k)| \leq \begin{cases} \dfrac{b-a}{8} \left\| f'' \right\|_1, \\ \text{for } f'' \in L_1[a,b],\ k=1, \\ . \\ \dfrac{(b-a)^3}{384} \left\| f^{iv} \right\|_1, \\ \text{for } f^{iv} \in L_1[a,b],\ k=2, \end{cases} \tag{5}$$

and for odd n

$$|T(a,b,2k+1)| \leq \begin{cases} \dfrac{1}{2} \left\| f' \right\|_1, \\ \text{for } f' \in L_1[a,b],\ k=0, \end{cases} \tag{6}$$

where $B(x,y)$ *is the Beta function and* $\|f^n\|_p$ *and* $\|f^n\|_\infty$ *are defined in above Theorem* 3.

2 Main results

Let the weight $w : [a, b] \to [0, \infty)$, be non-negative and integrable, i.e.,

$$\int_a^b w(t)\, dt < \infty.$$

The domain of w is finite. We denote the zero moment as

$$m(a, b) = \int_a^b w(t) dt.$$

The weighted norm of differentiable function whose derivatives belong to $\mathbf{L}_p(a, b)$ is defined as

$$\| \phi\|_{\omega,p} = \left(\int_a^b |\omega(t)\phi(t)|^p \, dt \right)^{1/p}.$$

Then the following inequality holds:

Theorem 2.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a differentiable mapping on (a, b), whose first derivative i.e., $f' : [a, b] \longrightarrow \mathbb{R}$, belongs to $L_p(a, b)$. Then, we have the inequality*

$$(7) \qquad \left| f(x) - \frac{1}{m(a, b)} \int_a^b w(t) f(t)\, dt \right| \leq$$

$$\leq \frac{(x - a)^{1+1/q} + (b - x)^{1+1/q}}{m(a, b)\,(q + 1)^{1/q}} \|f'\|_{w,p}.$$

Proof. Let us consider the kernel $k(., .) : [a, b]^2 \to \mathbb{R}$ given by

$$p(x, t) = \begin{cases} \int_a^t w(u) du, & \text{if } t \in [a, x] \\ \int_b^t w(u) du, & \text{if } t \in (x, b]. \end{cases}$$

The following weighted integral identity is proved in [1, *p*. 319],

$$f(x) - \frac{1}{m(a,b)} \int_a^b w(t) f(t)\, dt = \frac{1}{m(a,b)} \int_a^b p(x,t) f'(t)\, dt.$$

We have

$$\left| f(x) - \frac{1}{m(a,b)} \int_a^b w(t) f(t)\, dt \right| \le \frac{1}{m(a,b)} \int_a^b |p(x,t)|\, |f'(t)|\, dt. \tag{8}$$

Consider

$$\int_a^b |p(x,t)|\, |f'(t)|\, dt = \tag{9}$$

$$= \int_a^x \left(\int_a^t w(u)\, du \right) |f'(t)|\, dt + \int_x^b \left(\int_t^b w(u)\, du \right) |f'(t)|\, dt =$$

$$= \int_a^x (t-a) w(t)\, |f'(t)|\, dt + \int_x^b (b-t) w(t)\, |f'(t)|\, dt \le$$

$$\le \left(\int_a^x (t-a)^q dt \right)^{1/q} \left(\int_a^x |w(t)\, f'(t)|^p\, dt \right)^{1/p} +$$

$$+ \left(\int_x^b (b-t)^q dt \right)^{1/q} \left(\int_x^b w^p(t)\, |f'(t)|^p\, dt \right)^{1/p} =$$

$$= \left(\frac{(x-a)^{q+1}}{q+1} \right)^{1/q} \|f'\|_{w,p,[a,x]} + \left(\frac{(b-x)^{q+1}}{q+1} \right)^{1/q} \|f'\|_{w,p,(x,b]} \le$$

$$\le \frac{(x-a)^{1+1/q} + (b-x)^{1+1/q}}{(q+1)^{1/q}} \|f'\|_{w,p,[a,b]} =$$

$$= \frac{(x-a)^{1+1/q} + (b-x)^{1+1/q}}{(q+1)^{1/q}} \|f'\|_{w,p}.$$

From (8) and (9), we have the desired inequality (7).

Corollary 2.1. *Under the assumption of Theorem* 5*, we have the weighted mid-point inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{m(a,b)} \int_a^b w(t) f(t)\, dt \right| \leq \tag{10}$$

$$\leq \frac{(b-a)^{1+1/q}}{2^{1/q}(q+1)^{1/q} m(a,b)} \|f'\|_{w,p}.$$

Proof. This follows by the inequality (2.1), choosing $x = \frac{a+b}{2}$.

Remark 2.1. *For* $m(a,b) > 1$*, the result given in* (2.4) *is better than the comparable results available in the literature.*

Corollary 2.2. *Under the assumption of Theorem* 5*, we have the following weighted trapezoidal like inequality*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{m(a,b)} \int_a^b w(t) f(t)\, dt \right| \leq \tag{11}$$

$$\leq \frac{(b-a)^{1+1/q}}{(q+1)^{1/q} m(a,b)} \|f'\|_{w,p}.$$

Proof. This follows using (2.1) with $x = a$, $x = b$ adding the results and using the triangular inequality for the modulus.

Remark 2.2. *For* $m(a,b) > 1$*, the result given in* (2.5) *is better than the comparable results available in the literature.*

Corollary 2.3. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be defined in Theorem 5 and $f' \in L_2(a, b)$. Then, we have the inequality*

$$\left| f(x) - \frac{1}{m(a,b)} \int_a^b w(t) f(t)\, dt \right| \leq \tag{12}$$

$$\leq \frac{(x-a)^{3/2} + (b-x)^{3/2}}{\sqrt{3} m(a,b)} \|f'\|_{w,2}.$$

Proof. Apply inequality (2.1) for $p = q = 2$, we get the desired inequality (2.6).

Remark 2.3. *Taking into account the fact that the mapping*

$$h : [a, b] \longrightarrow \mathbb{R}, \; h(x) = (x-a)^{1+1/q} + (b-x)^{1+1/q},$$

has the property that

$$\underset{x \in (a,b)}{inf} \; h(x) = h(\frac{a+b}{2}) = \frac{(b-a)^{1+1/q}}{2^{\frac{1}{q}}},$$

and

$$\sup_{x \in (a,b)} h(x) = h(a) = h(b) = (b-a)^{1+1/q}.$$

We can get the best estimation from the inequality (2.1), *only when $x = \frac{a+b}{2}$, this yields the inequality* (2.4). *It shows that mid point estimation is better than the trapezoidal type estimation. Hence for $m(a, b) > 1$, the result given in* (2.4) *is better than the comparable results available in the literature.*

3 Applications in Numerical Integration

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, be a division of the interval $[a, b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, 1, \cdots, n-1)$, a sequence of intermediate points and $h_i = x_{i+1} - x_i$ $(i = 0, 1, \cdots, n-1)$. We have the following quadrature formula:

Theorem 3.1. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a differentiable mapping on $[a,b]$ whose first derivative i.e., $f' : (a,b) \longrightarrow \mathbb{R}$, belongs to $L_p(a,b)$. Then, we have the following quadrature formula:*

$$\int_a^b w(t) f(t)\, dt = A(f, f', \xi, I_n) + R(f, f', \xi, I_n),$$

where

$$A(f, f', \xi, I_n) = \sum_{i=0}^{n-1} m(x_i, x_{i+1}) f(\xi_i),$$

and the remainder satisfies the estimation

$$|R(f, f', \xi, I_n)| \leq \frac{\|f'\|_{w,p}}{(q+1)^{1/q}} \sum_{i=0}^{n-1} \left[(\xi_i - x_i)^{1+1/q} + (x_{i+1} - \xi_i)^{1+1/q} \right], \tag{13}$$

for all ξ_i.

Proof. Applying Theorem 5 on the interval $[x_i, x_{i+1}]$ $(i = 0, 1, \cdots, n-1)$, and summing over i from $i = 0$ to $n-1$ and using the generalized triangular inequality, we deduce the desired estimation (3.1).

Remark 3.1. *If we choose $\xi_i = \frac{x_i + x_{i+1}}{2}$, we recapture the weighted mid-point quadrature formula*

$$\int_a^b w(t) f(t)\, dt = A_M + R_M,$$

where the remainder R_M satisfies the estimation:

$$|R_M| \leq \frac{\|f'\|_{w,p}}{2^{1/q}(q+1)^{1/q}} \sum_{i=0}^{n-1} h_i^{1+1/q}. \tag{14}$$

Remark 3.2. *To derive the corresponding results for the euclidean norm* $\|f'\|_{w,2}$, *we put* $p = q = 2$, *in* (3.2).

Remark 3.3. *The corresponding quadrature formulas for equidistant partitioning can be obtained by choosing* $x_i = a + i\frac{b-a}{n}$ $(i = 0, 1, \cdots, n-1)$.

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