General Mathematics Vol. 14, No. 4 (2006), 15–28

Simultaneous Approximation By Two Dimensional Hybrid Positive Linear Operators [1]

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Abstract

In this paper, we obtain some simultaneous approximation properties and asymptotic formulas of two dimensional hybrid (Szász-Mirakian and Lupaş Durrmeyer) positive linear operators and their partial derivatives.

2000 Mathematics Subject Classification: 41A28, 41A36.

Key Words and Phrases: Szász-Mirakian operator, Lupaş operator, Simultaneous approximation.

[1] *Received 1 October, 2006*

Accepted for publication (in revised form) 9 November, 2006

1 Introduction

To approximate Lebesgue integrable functions on interval $[0,\infty)$, Gupta and Srivastava [6] proposed a sequence of linear positive operators, by combining the well known Szász-Mirakian operator with the weight function of Lupaş operator defined as:

$$(S_n f)(x) = (n-1)\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} v_{n,k}(t)f(t)dt, \quad x \in R = [0,\infty), \tag{1}$$

where

$$b_{n,k}(x) = \frac{e^{-nx}(nx)^k}{k!}, \quad v_{n,k}(t) = \left(\begin{array}{c} n+k-1 \\ k \end{array}\right)\frac{t^k}{(1+t)^{n+k}}$$

and $n \in N = \{1, 2, ...\}$.

Now we consider two dimensional hybrid positive linear operators as:

$$M_n^{[i,j]}(f; x,y) = (n-1)^2\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{n,k}^{(i)}(x)b_{n,l}^{(j)}(y)\int_0^{\infty}\int_0^{\infty} v_{n,k}(s)v_{n,l}(t)f(s,t)dsdt, \tag{2}$$

where

$$(x,y) \in [0,\infty)\times[0,\infty) \text{ and } f \in C([0,\infty)\times[0,\infty)).$$

In particular, if $f^{(0)}(x) = f(x)$ then the meanings of $b_n^{[i,0]}(f;x,y)$ and $b_n^{[0,j]}(f;x,y)$ are clear.

By $H[0,\infty)^2$, we denote the class of all measurable functions defined on $[0,\infty)$ satisfying

$$\int_0^{\infty}\int_0^{\infty}\frac{|f(s)|\,|f(t)|}{\{(1+s)(1+t)\}^{n+1}}dsdt < \infty, \text{ for some positive integer n.}$$

This class is bigger than the class of all Lebesgue integrable functions on $[0, \infty)$.

Very recently author [1] has studied simultaneous approximation for two dimensional Lupaş-Durrmeyer operators and in [2], he studied simultaneous approximation for one variable. Gupta and other researchers also studied simultaneous approximation for one as well as two variables for similar type operators (see e.g. [3], [4], [5], [8]).

The main object of this paper is to obtain the properties of simultaneous approximation by two dimensional Szász-Mirakian-Lupaş-Durrmeyer operators and obtained several asymptotic formulae for the partial derivative of these operators (2).

2 Auxiliary Results

In this section, we shall mention certain results which are necessary to prove our main theorem.

Lema 2.1.*[7] For $m \in N \cup \{0\}$, if we define*

$$V_{n,m}(x) = \sum_{k=0}^{\infty} b_{n,k}(x) \left(\frac{k}{n} - x \right)^m ,$$

then

$$nV_{n,m+1}(x) = x \left[V'_{n,m}(x) + mV_{n,m-1}(x) \right] .$$

Consequently, we have

(i) *$V_{n,m}(x)$ is a polynomial in x of degree $\leq m$.*

(ii) *$V_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$, where $[\gamma]$ denotes the integral part of γ.*

Lema 2.2. *[6, 7] There exists the polynomials $\varphi_{c,h,r}(x)$ independent of n and k such that*

$$x^r \frac{d^r}{dx^r}[e^{-nx}(nx)^k] = \sum_{\substack{2c+h\le r\\ c,h\ge 0}} n^c(k-nx)^h \varphi_{c,h,r}(x)[e^{-nx}(nx)^k]. \tag{3}$$

Lema 2.3. *If $f(x,y)$ is differentiable r_1+r_2 times on $[0,\infty)$, then we get*

$$M_n^{[r_1,r_2]}(f;x,y) = \frac{n^{r_1+r_2}(n-r_1-1)!(n-r_2-1)!}{\{(n-2)!\}^2} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{n,k}(x)b_{n,l}(y)\cdot$$

$$\cdot \int_0^\infty \int_0^\infty v_{n-r_1,k+r_1}(s)v_{n-r_2,l+r_2}(t)\frac{\partial^{r_1+r_2}}{\partial s^{r_1}\partial t^{r_2}} f(s,t)\, dsdt.$$

Proof. From (2), we have

$$M_n^{[r_1,r_2]}(f;x,y) =$$

$$= (n-1)^2 \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{n,k}^{(r_1)}(x) b_{n,l}^{(r_2)}(y) \int_0^\infty \int_0^\infty v_{n,k}(s)v_{n,l}(t) f(s,t) dsdt$$

Using Leibnitz theorem, we get

$$M_n^{[r_1,r_2]}(f;x,y) = (n-1)^2 \sum_{i=0}^{r_2}\sum_{j=0}^{r_2}\sum_{k=i}^{\infty}\sum_{l=j}^{\infty} n^{r_1+r_2}(-1)^{r_1-i}(-1)^{r_2-j}\binom{r_1}{i}\binom{r_2}{j}\cdot$$

$$\cdot\frac{e^{-nx}(nx)^{k-i}}{(k-i)!} \quad \cdot\frac{e^{-ny}(ny)^{l-j}}{(l-j)!} \int_0^\infty \int_0^\infty v_{n,k}(s)v_{n,l}(t) f(s,t)\, dsdt =$$

$$= (n-1)^2 n^{r_1+r_2} \sum_{i=0}^{r_2}\sum_{j=0}^{r_2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} (-1)^{r_1-i}(-1)^{r_2-j}\binom{r_1}{i}\binom{r_2}{j} b_{n,k}(x)b_{n,l}(y)\cdot$$

$$\cdot\int_0^\infty \int_0^\infty v_{n,k+i}(s)v_{n,l+j}(t) f(s,t)\, dsdt = (n-1)^2 n^{r_1+r_2} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{n,k}(x)b_{n,l}(y)\cdot$$

$$\cdot\int_0^\infty \int_0^\infty \sum_{i=0}^{r_2}\sum_{j=0}^{r_2} (-1)^{r_1-i}(-1)^{r_2-j}\binom{r_1}{i}\binom{r_2}{j} v_{n,k+i}(s)v_{n,l+j}(t) f(s,t)\, dsdt.$$

Once again applying Leibnitz theorem, we obtain

$$v^{(q)}_{n-q,u+q}(z) = \frac{(n-1)!}{(n-q-1)!}\sum_{w=0}^{q}(-1)^w \begin{pmatrix} q \\ w \end{pmatrix} v_{n,u+w}(z),$$

where $q = r_1, r_2;\ \ w = i, j;\ \ u = k, l$. So we have

$$M_n^{[r_1,r_2]}(f;x,y) = \frac{n^{r_1+r_2}(n-r_1-1)!(n-r_2-1)!}{\{(n-2)!\}^2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{n,k}(x)b_{n,l}(y)\cdot$$

$$\cdot\int_0^\infty\int_0^\infty (-1)^{r_1+r_2} v^{(r_1)}_{n-r_1,k+r_1}(s) v^{(r_2)}_{n-r_2,l+r_2}(t) f(s,t)\, ds dt.$$

Further integrating by parts $r_1 + r_2$ times, we get the required result.

Lema 2.4. *Let the $i-th$ order moment be defined by*

$$\mu_{n,r,i}(x) = (n-r-1)\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^\infty v_{n-r,k+r}(s)(t-x)^i ds$$

then, we have the following recurrence relation:

$$(n-i-r-2)\mu_{n,r,i+1}(x) = x\mu'_{n,r,i}(x) + \left[(i+1)(1+2x) + r(1+x)\right]\mu_{n,r,i}(x)$$
$$+\, ix(x+2)\mu_{n,r,i-1}(x), \tag{4}$$

where $n > i + r + 2$. Consequently,

(i) *we have*

$$\mu_{n,r,0}(x) = 1,\ \ \mu_{n,r,1}(x) = \frac{r(x+1) + (1+2x)}{(n-r-2)}$$

and

$$\mu_{n,r,2}(x) = \frac{x^2(n+r^2+5r+6) + 2x(n+r^2+4r+3) + (r^2+3r+2)}{(n-r-2)(n-r-3)},$$

(ii) *for all $x \in [0,\infty)$, we get $\mu_{n,r,i}(x) = O\left(n^{-[(i+1)/2]}\right)$.*

Proof. First we prove (4), by using

$$xb'_{n,k}(x) = (k - nx)b_{n,k}(x) \text{ and } t(1+t)v'_{n,k}(t) = (k - nt)v_{n,k}(t).$$

We have from the definition of $\mu_{n,r,i}(x)$

$$x\mu'_{n,r,i}(x) = (n-r-1)\sum_{k=0}^{\infty} xb'_{n,k}(x)\int_0^{\infty} v_{n-r,k+r}(t)(t-x)^i dt - ix\mu_{n,r,i-1}(x).$$

Therefore, we have

$$x\left[\mu'_{n,r,i}(x) + i\mu_{n,r,i-1}(x)\right] =$$

$$= (n-r-1)\sum_{k=0}^{\infty}(k-nx)b_{n,k}(x)\int_0^{\infty} v_{n-r,k+r}(t)(t-x)^i dt$$

$$= (n-r-1)\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty}\left[(k+r)-(n-r)t\right]v_{n-r,k+r}(t)(t-x)^i dt +$$

$$+(n-r)\mu_{n,r,i+1}(x) + (n-r)x\mu_{n,r,i}(x) - (nx+r)\mu_{n,r,i}(x) =$$

$$= (n-r-1)\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} t(1+t)v'_{n-r,k+r}(t)(t-x)^i dt + (n-r)\mu_{n,r,i+1}(x) -$$

$$-(1+x)r\mu_{n,r,i}(x) =$$

$$= (n-r-1)\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty}\left[(1+2x)(t-x)+(t-x)^2+x(1+x)\right]\cdot$$

$$\cdot b'_{n-r,k+r}(t)(t-x)^i dt + (n-r)\mu_{n,r,i+1}(x) - (1+x)r\mu_{n,r,i}(x) =$$

$$= -(i+1)(1+2x)\mu_{n,r,i}(x) - (i+2)\mu_{n,r,i+1}(x) - x(1+x)i\mu_{n,r,i-1}(x) +$$

$$+(n-r)\mu_{n,r,i+1}(x) - (1+x)r\mu_{n,r,i}(x).$$

This leads to the proof of (4).

Lema 2.5. *Suppose that*

$$B_{n,r_1,r_2,i,j}(x,y) = (n-r_1-1)(n-r_2-1)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} b_{n,k}(x)b_{n,l}(y)\cdot$$
$$\cdot \int_0^{\infty}\int_0^{\infty} v_{n-r_1,k+r_1}(s)v_{n-r_2,l+r_2}(t)\,(s-x)^i\,(t-y)^j\,dsdt =$$
$$= \mu_{n,r_1,i}(x).\mu_{n,r_2,j}(y), \tag{5}$$

then we obtain the following results by Lemma 2.4

$$B_{n,r_1,r_2,0,0}(x,y) = 1, B_{n,r_1,r_2,i,j}(x,y) = O\left(n^{-\left(\left[\frac{i+1}{2}\right]+\left[\frac{j+1}{2}\right]\right)}\right),$$

$$B_{n,r_1,r_2,1,0}(x,y) = \frac{r_1(1+x)+(1+2x)}{(n-r_1-2)},$$

$$B_{n,r_1,r_2,0,1}(x,y) = \frac{r_2(1+y)+(1+2y)}{(n-r_2-2)},$$

$$B_{n,r_1,r_2,1,1}(x,y) = \frac{\{r_1(1+x)+(1+2x)\}\{r_2(1+y)+(1+2y)\}}{(n-r_1-2)(n-r_2-2)},$$

$$B_{n,r_1,r_2,2,0}(x,y) = \frac{x^2(n+r_1^2+5r_1+6)+2x(n+r_1^2+4r_1+3)+(r_1^2+3r_1+2)}{(n-r_1-2)(n-r_1-3)},$$

$$B_{n,r_1,r_2,0,2}(x,y) = \frac{y^2(n+r_2^2+5r_2+6)+2y(n+r_2^2+4r_2+3)+(r_2^2+3r_2+2)}{(n-r_2-2)(n-r_2-3)}.$$

3 Main Results

Now we consider slightly modified operators M_n^*, for our convenience,

$$M_n^{*[r_1,r_2]}f = \frac{1}{\mathrm{C}_1(n,r_1,r_2)} M_n^{[r_1,r_2]}f,$$

where M_n preserve constants and

$$\mathrm{C}_1(n,r_1,r_2) = \frac{n^{r_1+r_2}(n-r_1-2)!(n-r_2-2)!}{\{(n-2)!\}^2}.$$

Theorem 3.1. *Suppose f is bounded on every finite subinterval of $[0,\infty)$ and $f \in H[0,\infty)^2$. If $f^{(r+2)}$ exists at a fixed point $x \in [0,\infty)$ and $\left|\frac{\partial^{r+2}}{\partial x^j \partial y^{r+2-j}} f(x,y)\right| \leq \mu x^\alpha y^\beta,\ (x \to \infty,\ y \to \infty);\quad j = 1,...,r+2$ for some $\alpha, \beta \geq 0$, then we get*

$$\lim_{n\to\infty}\left[M_n^{*[r,0]}(f;x,y) - \frac{\partial^r}{\partial x^r} f(x,y)\right] = \tag{6}$$

$$= (1+2y)\frac{\partial^{r+1}}{\partial x^r \partial y} f(x,y) + \{r(1+x)+(1+2x)\}\frac{\partial^{r+1}}{\partial x^{r+1}} f(x,y) +$$

$$+ y(2+y)\frac{\partial^{r+2}}{\partial x^r \partial y^2} f(x,y) + \frac{x(2+x)}{2}\frac{\partial^{r+2}}{\partial x^{r+2}} f(x,y).$$

and

$$\lim_{n\to\infty}\left[M_n^{*[0,r]}(f;x,y) - \frac{\partial^r}{\partial y^r} f(x,y)\right] = \tag{7}$$

$$= (1+2x)\frac{\partial^{r+1}}{\partial y^r \partial x} f(x,y) + \{r(1+y)+(1+2y)\}\frac{\partial^{r+1}}{\partial y^{r+1}} f(x,y) +$$

$$+ x(2+x)\frac{\partial^{r+2}}{\partial y^r \partial x^2} f(x,y) + \frac{y(2+y)}{2}\frac{\partial^{r+2}}{\partial y^{r+2}} f(x,y).$$

Proof. Since the proof of (7) is identical therefore we shall give the prove (6) only. By Taylorş expansion of $f(s,t)$, we have

$$f(s,t) = \sum_{d=0}^{r+2} \sum_{i+j=d} \frac{1}{i!j!}\left(\frac{\partial^d}{\partial x^i \partial y^j} f(x,y)\right)(s-x)^i (t-y)^j +$$

$$+ \sum_{i+j=r+2} \varepsilon(s,t,x,y)(s-x)^i (t-y)^j.$$

where $\varepsilon(s,t,x,y) \to 0$ as $s \to x,\ \ t \to y$ and $\varepsilon(s,t,x,y) \leq \mu(s-x)^\alpha (t-y)^\beta$ as $s \to \infty$, $x \to \infty$ for some $\alpha, \beta > 0$ then

$$n\left[M_n^{*[r,0]}(f;x,y) - \frac{\partial^r}{\partial x^r} f(x,y)\right] =$$

$$= n\sum_{d=0}^{r+2}\sum_{i+j=d}\frac{1}{i!j!}\left(\frac{\partial^d}{\partial x^i\partial y^j}f(x,y)\right)M_n^{*[r,0]}\left((s-x)^i(t-y)^j;x,y\right)+$$

$$+n\sum_{i+j=r+2}M_n^{*[r,0]}\left(\varepsilon(s,t,x,y)(s-x)^i(t-y)^j;x,y\right)-n\frac{\partial^r}{\partial x^r}f(x,y)=$$

$$=Q_1+Q_2-n\frac{\partial^r}{\partial x^r}f(x,y).$$

From Lemma 2.3, we get

$$Q_1=n\sum_{d=0}^{r+2}\sum_{i+j=d}\frac{1}{i!j!}\frac{\partial^d}{\partial x^i\partial y^j}f(x,y)(n-1)(n-r-1)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}b_{n,k}(x)b_{n,l}(y)\cdot$$

$$\cdot\int_0^\infty\int_0^\infty v_{n-r,k+r}(s)v_{n,l}(t)\frac{\partial^r}{\partial s^r}\left((s-x)^i(t-y)^j\right)dsdt=$$

$$=\frac{n}{r!}\frac{\partial^r}{\partial x^r}f(x,y)(n-1)(n-r-1)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}b_{n,k}(x)b_{n,l}(y)\cdot$$

$$\cdot\int_0^\infty\int_0^\infty v_{n-r,k+r}(s)v_{n,l}(t)r!dsdt+$$

$$+\frac{n}{r!1!}\frac{\partial^{r+1}}{\partial x^r\partial y}f(x,y)(n-1)(n-r-1)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}b_{n,k}(x)b_{n,l}(y)\cdot$$

$$\cdot\int_0^\infty\int_0^\infty v_{n-r,k+r}(s)v_{n,l}(t)r!(t-y)dsdt+$$

$$+\frac{n}{(r+1)!}\frac{\partial^{r+1}}{\partial x^{r+1}}f(x,y)(n-1)(n-r-1)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}b_{n,k}(x)b_{n,l}(y)\cdot$$

$$\cdot\int_0^\infty\int_0^\infty v_{n-r,k+r}(s)v_{n,l}(t)\frac{\partial^r}{\partial s^r}(s-x)^{r+1}dsdt+$$

$$+\frac{n}{r!2!}\frac{\partial^{r+2}}{\partial x^r\partial y^2}f(x,y)(n-1)(n-r-1)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}b_{n,k}(x)b_{n,l}(y)\cdot$$

$$\cdot\int_0^\infty\int_0^\infty v_{n-r,k+r}(s)v_{n,l}(t)r!(t-y)^2dsdt+$$

$$+\frac{n}{(r+1)!}\frac{\partial^{r+2}}{\partial x^{r+1}\partial y}f(x,y)(n-1)(n-r-1)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}b_{n,k}(x)b_{n,l}(y)\cdot$$

$$\cdot\int_0^\infty\int_0^\infty v_{n-r,k+r}(s)v_{n,l}(t)\frac{\partial^r}{\partial s^r}\left\{(s-x)^{r+1}(t-y)\right\}dsdt+$$

$$+\frac{n}{(r+2)!}\frac{\partial^{r+2}}{\partial x^{r+2}}f(x,y)(n-1)(n-r-1)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}b_{n,k}(x)b_{n,l}(y)\cdot$$

$$\cdot\int_0^\infty\int_0^\infty v_{n-r,k+r}(s)v_{n,l}(t)\frac{\partial^r}{\partial s^r}(s-x)^{r+2}dsdt=$$

$$=n\frac{\partial^r}{\partial x^r}f(x,y)B_{n,r,0,0,0}(x,y)+n\frac{\partial^{r+1}}{\partial x^r\partial y}f(x,y)B_{n,r,0,0,1}(x,y)+$$

$$+n\frac{\partial^{r+1}}{\partial x^{r+1}}f(x,y)B_{n,r,0,1,0}(x,y)+\frac{n}{2}\frac{\partial^{r+2}}{\partial x^r\partial y^2}f(x,y)B_{n,r,0,0,2}(x,y)+$$

$$+n\frac{\partial^{r+2}}{\partial x^{r+1}\partial y}f(x,y)B_{n,r,0,1,1}(x,y)+\frac{n}{2}\frac{\partial^{r+2}}{\partial x^{r+2}}f(x,y)B_{n,r,0,2,0}(x,y)=$$

$$=n\frac{\partial^r}{\partial x^r}f(x,y)+\frac{n(1+2y)}{(n-2)}\frac{\partial^{r+1}}{\partial x^r\partial y}f(x,y)+\frac{n\{r(1+x)+(1+2x)\}}{(n-r-2)}\frac{\partial^{r+1}}{\partial x^{r+1}}f(x,y)+$$

$$+\frac{n\{y^2(n+6)+2y(n+3)+2\}}{(n-2)(n-3)}\frac{\partial^{r+2}}{\partial x^r\partial y^2}f(x,y)+$$

$$+\frac{n(1+2y)\{r(1+x)+(1+2x)\}}{(n-2)(n-r-2)}\frac{\partial^{r+2}}{\partial x^{r+1}\partial y}f(x,y)+$$

$$+\frac{n\{x^2(n+r^2+5r+6)+2x(n+r^2+4r+3)+(r^2+3r+2)\}}{2(n-r-2)(n-r-3)}\frac{\partial^{r+2}}{\partial x^{r+2}}f(x,y),$$

by Lemma 2.4, we obtain the above results. In order to prove the theorem, it is sufficient to show that

$$E_n\cong x^rQ_2=n(n-1)^2\sum_{i+j=r+2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}x^rb_{n,k}^{(r)}(x)b_{n,l}(y)\cdot$$

$$\cdot\int_0^\infty\int_0^\infty v_{n,k}(s)v_{n,l}(t)\varepsilon(s,t,x,y)(s-x)^i(t-y)^jdsdt\to 0\ \text{ as }\ (n\to\infty).$$

Using Lemma 2.2, we get

$$|E_n| \leq n(n-1)^2 \sum_{i+j=r+2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{2c+h\leq r} n^c |k-nx|^h |\varphi_{c,h,r}(x)| b_{n,k}(x) b_{n,l}(y) \cdot$$

$$\cdot \int_0^{\infty} \int_0^{\infty} v_{n,k}(s) v_{n,l}(t) |\varepsilon(s,t,x,y)| |s-x|^i |t-y|^j ds dt \leq$$

$$\leq n\chi(x) \sum_{i+j=r+2} \sum_{2c+h\leq r} n^c \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left((b_{n,k}(x) b_{n,l}(y))^{\frac{1}{2}} |k-nx|^h \right) \cdot$$

$$\cdot (b_{n,k}(x) b_{n,l}(y))^{\frac{1}{2}} (n-1)^2 \int_0^{\infty} \int_0^{\infty} v_{n,k}(s) v_{n,l}(t) |\varepsilon(s,t,x,y)| |s-x|^i |t-y|^j ds dt \leq$$

$$\leq n\chi(x) \sum_{i+j=r+2} \sum_{2c+h\leq r} n^c \left[\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{n,k}(x) b_{n,l}(y) (k-nx)^{2h} \right]^{\frac{1}{2}} \cdot$$

$$\cdot \left[\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{n,k}(x) b_{n,l}(y) \cdot \right.$$

$$\left. \cdot \left\{ (n-1)^2 \int_0^{\infty} \int_0^{\infty} v_{n,k}(s) v_{n,l}(t) |\varepsilon(s,t,x,y)| |s-x|^i |t-y|^j ds dt \right\}^2 \right]^{\frac{1}{2}},$$

where

$$\chi(x) = \sum_{\substack{2c+h\leq r \\ c,h\geq 0}} \sup |\varphi_{c,h,r}(x)| .$$

From Lemma 2.1, we have

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{n,k}(x) b_{n,l}(y) (k-nx)^{2h} = n^{2h} \sum_{k=0}^{\infty} b_{n,k}(x) \left(\frac{k}{n} - x \right)^{2h} =$$

$$= n^{2h} O\left(n^{-\left[\frac{2h+1}{2}\right]} \right) = n^{2h} O\left(n^{-h} \right) = n^h O(1)$$

and let

$$\rho_n = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} b_{n,k}(x) b_{n,l}(y) \cdot$$

$$\cdot\left[(n-1)^2\int_0^\infty\int_0^\infty v_{n,k}(s)v_{n,l}(t)|\varepsilon(s,t,x,y)(s-x)^i(t-y)^j|dsdt\right]^2.$$

Therefore, we obtain

$$|E_n|\le n\chi(x)\sum_{i+j=r+2}\sum_{2c+h\le r} n^c\left(n^hO(1)\right)^{\frac{1}{2}}(\rho_n)^{\frac{1}{2}}.$$

Now

$$\left[(n-1)^2\int_0^\infty\int_0^\infty v_{n,k}(s)v_{n,l}(t)|\varepsilon(s,t,x,y)(s-x)^i(t-y)^j|dsdt\right]^2\le$$

$$\le(n-1)^2\int_0^\infty\int_0^\infty v_{n,k}(s)v_{n,l}(t)dsdt\cdot$$

$$\cdot(n-1)^2\int_0^\infty\int_0^\infty v_{n,k}(s)v_{n,l}(t)\varepsilon^2(s,t,x,y)(s-x)^{2i}(t-y)^{2j}dsdt=$$

$$=(n-1)^2\int_0^\infty\int_0^\infty v_{n,k}(s)v_{n,l}(t)\varepsilon^2(s,t,x,y)(s-x)^{2i}(t-y)^{2j}dsdt=$$

$$=(n-1)^2\left[\int_{(s-x)^2+(t-y)^2\le\delta^2}+\int_{(s-x)^2+(t-y)^2>\delta^2}\right]\cdot$$

$$\cdot v_{n,k}(s)v_{n,l}(t)\varepsilon^2(s,t,x,y)(s-x)^{2i}(t-y)^{2j}dsdt.$$

For a given $\eta>0$, there exists a $\delta>0$ such that $|\varepsilon(s,t,x,y)|<\eta$ whenever $(s-x)^2+(t-y)^2\le\delta^2$. For $(s-x)^2+(t-y)^2>\delta^2$, we obtain $|\varepsilon(s,t,x,y)|<K(s-x)^\alpha(t-y)^\beta$.

$$\rho_n=\eta^2\sum_{k=0}^\infty\sum_{l=0}^\infty b_{n,k}(x)b_{n,l}(y)(n-1)^2\cdot$$

$$\cdot\int_0^\infty\int_0^\infty v_{n,k}(s)v_{n,l}(t)(s-x)^{2i}(t-y)^{2j}dsdt+$$

$$+K\sum_{k=0}^\infty\sum_{l=0}^\infty b_{n,k}(x)b_{n,l}(y)(n-1)^2\int_{(s-x)^2+(t-y)^2>\delta^2}\frac{(s-x)^2+(t-y)^2}{\delta^2}\cdot$$

$$\cdot(s-x)^{2(i+\alpha)}(t-y)^{2(j+\beta)}v_{n,k}(s)v_{n,l}(t)dsdt=$$

$$= \eta^2 O\left(n^{-\left(\left[\frac{2i+1}{2}\right]+\left[\frac{2i+1}{2}\right]\right)}\right) + \frac{1}{\delta^2} O\left(n^{-\left(\left[\frac{2i+2\alpha+2+1}{2}\right]+\left[\frac{2j+2\beta+1}{2}\right]\right)}\right) +$$

$$+\frac{1}{\delta^2} O\left(n^{-\left(\left[\frac{2i+2\alpha+1}{2}\right]+\left[\frac{2j+2\beta+2+1}{2}\right]\right)}\right) =$$

$$= \eta^2 O\left(n^{-(i+j)}\right) + \frac{1}{\delta^2} O\left(n^{-(i+j)} n^{-\left(\left[\frac{2\alpha+1}{2}\right]+1+\left[\frac{2\beta+1}{2}\right]\right)}\right) +$$

$$+\frac{1}{\delta^2} O\left(n^{-(i+j)} n^{-\left(\left[\frac{2\alpha+1}{2}\right]+1+\left[\frac{2\beta+1}{2}\right]\right)}\right) =$$

$$= O\left(n^{-(i+j)}\right)\left(\eta^2 + \frac{2}{\delta^2} n^{-\zeta}\right), \quad \text{where } \zeta = \left[\frac{2\alpha+1}{2}\right] + 1 + \left[\frac{2\beta+1}{2}\right] > 0.$$

Thus, we get

$$|E_n| \le n\chi(x) \sum_{i+j=r+2} \sum_{2c+h\le r} n^c \left[n^h O(1)\right]^{\frac{1}{2}} \left[O\left(n^{-(i+j)}\right)\left(\eta^2 + \frac{2}{\delta^2} n^{-\zeta}\right)\right]^{\frac{1}{2}} \le$$

$$\le n\chi(x) \sum_{2c+h\le r} n^c \left(n^h\right)^{\frac{1}{2}} O(1) \sum_{i+j=r+2} \left[O\left(n^{-(i+j)}\right)\left(\eta^2 + \frac{2}{\delta^2} n^{-\zeta}\right)\right]^{\frac{1}{2}} =$$

$$= O(1) n^{\frac{r+2}{2}} n^{-\frac{r+2}{2}} \left(\eta^2 + \frac{2}{\delta^2} n^{-\zeta}\right)^{\frac{1}{2}} =$$

$$= O(1)\left(\eta^2 + \frac{2}{\delta^2} n^{-\zeta}\right)^{\frac{1}{2}} \to 0, \quad \text{as } n \to \infty.$$

This completes the proof of (6).

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