General Mathematics Vol. 15, Nr. 2–3(2007), 77-87

First order strong differential superordination

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Abstract

The notion of differential superordination was introduced in [3] by S.S. Miller and P.T. Mocanu as a dual concept of differential subordination [2]. The notion of strong differential subordination was introduced by J.A. Antonino, S. Romaguera in [1]. The notion of strong differential superordination was introduced in [4] as a dual concept of strong differential subordination. In this paper we refer at the special case of first order strong differential superordinations.

2000 Mathematical Subject Classification: 30C45, 30A20, 34A30.

Key words and phrases: analytic function, differential subordination, differential superordination.

1 Introduction

Let Ω be any set in the complex plane $\mathbb{C}$, let p be analytic in the unit disk U and let $\psi(r, s, t; z, \xi) : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$.

In this article we consider the dual problem of determining properties of functions p that satisfy the strong differential superordination

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z, \xi) \mid z \in U,\ \xi \in \overline{U}\}.$$

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in U. For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H};\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots,\ z \in U\},$$

$$A_n = \{f \in A,\ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots,\ z \in U\},$$

with $A_1 = A$.

In addition, we need the classes of convex (univalent) functions given respectively by

$$K = \{f \in A,\ \text{Re}\ zf''(z)/f'(z) + 1 > 0\}$$

and

$$S^* = \{f \in A,\ \text{Re}\ zf'(z)/f(z) > 0\}.$$

For $0 < r < 1$, we let $U_r = \{z;\ |z| < r\}$.

In order to prove our main results, we use the following definitions and lemmas:

Definition 1. [2, p.24] *We denote by Q the set of functions f that are analytic and injective in $\overline{U} \setminus E(f)$, where*

$$E(f) = \left\{\zeta \in \partial U;\ \lim_{z \to \zeta} f(z) = \infty\right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$. The subclass of Q for which $f(0) = a$ is defined by $Q(a)$.

Lemma A. [3, Lemma A]. *Let* $p \in Q(a)$*, and let*

$$q(z) = a + a_n z^n + \dots$$

be analytic in U *with* $q(z) \neq a$ *and* $n \geq 1$*. If* q *is not subordinate to* p*, then there exists points* $z_0 = r_0 e^{i\theta_0} \in U$ *and* $\zeta_0 \in \partial U \setminus E(p)$*, and an* $m \geq n \geq 1$ *for which* $q(U_{r_0}) \subset p(U)$*,*

i) $q(z_0) = p(\zeta_0)$

ii) $z_0 q'(z_0) = m\zeta_0 p'(\zeta_0)$*.*

Lemma B. [2, Theorem 2.6.4, p.67] *Let* $f \in A$ *and* $L_\gamma : A \to A$ *is the integral operator defined by*

$$L_\gamma(f) = F(z) = \frac{\gamma+1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt, \quad \operatorname{Re}\gamma \geq 0$$

then

$$L_\gamma[K] \subset K.$$

Definition 2. [4, Definition 1] *Let* $H(z,\xi)$ *be analytic in* $U \times \overline{U}$ *and let* $f(z)$ *analytic and univalent in* U*. The function* $H(z,\xi)$ *is strongly subordinate to* $f(z)$*, or* $f(z)$ *is said to be strongly superordinate to* $H(z,\xi)$*, written* $f(z) \prec\prec H(z,\xi)$ *if for* $\xi \in \overline{U}$*, the function of* z*,* $H(z,\xi)$ *is subordinate to* $f(z)$*. If* $H(z,\xi)$ *is univalent, then* $f(z) \prec\prec H(z,\xi)$ *if and only if* $f(0) = H(0,\xi)$ *and* $f(U) \subset H(U \times \overline{U})$*.*

2 Main results

Definition 3. *Let* $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ *and let* h *be analytic in* U*. If* p *and* $\varphi(p(z), zp'(z); z, \xi)$ *are univalent in* U*, for all* $\xi \in \overline{U}$ *and satisfy the first* *order strong differential superordination*

$$h(z) \prec\prec \varphi(p(z), zp'(z); z, \xi) \tag{1}$$

then p is called a solution of the strong differential superordination. An analytic function q is called a subordinant of the solutions of the strong differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1) is said to be the best subordinant. Note that the best subordinant is unique up to a rotation of U. For Ω a set in $\mathbb{C}$, with φ and p as given in Definition 3, suppose (1) is replaced by

$$\Omega \subset \{\varphi(p(z), zp'(z); z, \xi) \mid z \in U,\ \xi \in \overline{U}\}. \tag{1'}$$

Definition 4. *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a,n]$ with $q'(z) \neq 0$. The class of admissible functions $\phi_n[\Omega, q]$, consists of those functions $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ that satisfy the*

$$\varphi(r, s;, \zeta, \xi) \in \Omega \tag{2}$$

whenever $r = q(z)$, $s = \dfrac{zq'(z)}{m}$, where $z \in U$, $\zeta \in \partial U$, $\xi \in \overline{U}$ and $m \geq n \geq 1$.

Theorem 1. *Let $\Omega \subset \mathbb{C}$, $q \in \mathcal{H}[a,n]$, $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$, and suppose that*

$$\varphi(q(z), tzq'(z); \zeta, \xi) \in \Omega \tag{3}$$

for $z \in U$, $\zeta \in \partial U$, $\xi \in \overline{U}$ and $0 < t < \dfrac{1}{n} \leq 1$. If $p \in Q(a)$ and $\varphi(p(z), zp'(z); z, \xi)$ is univalent in U, for all $\xi \in \overline{U}$, then

$$\Omega \subset \{\varphi(p(z), zp'(z); z, \xi),\ z \in U,\ \xi \in \overline{U}\} \tag{4}$$

implies

$$q(z) \prec p(z), \quad z \in U.$$

Proof. Assume q not subordinate to p. By Lemma A there exist points $z_0 = r_0 e^{i\theta_0} \in U$, and $\zeta_0 \in \partial U \setminus E(p)$, and an $m \geq n \geq 1$ that satisfy $q(z_0) = p(\zeta_0)$ and $z_0 q'(z_0) = m\zeta_0 p'(\zeta_0)$. Let $r = q(z_0) = p(\zeta_0)$, $s = \frac{z_0 q'(z_0)}{m} = \zeta_0 p'(\zeta_0)$ and $\zeta = \zeta_0$ in Definition 4 and using (3) we obtain

$$\varphi(p(\zeta_0), \zeta_0 p'(\zeta_0); \zeta_0, \xi) \in \Omega. \tag{5}$$

Since ζ_0 is a boundary point we deduce that (5) contradicts (4) and we must $q(z) \prec p(z)$, $z \in U$.

We next consider the special situation when h is analytic on U and $h(U) = \Omega \neq \mathbb{C}$. In this case, the class $\phi_n[h(U), q]$ is written as $\phi_n[h, q]$ and the following result is an immediate consequence of Theorem 1.

Theorem 2. *Let h be analytic in U, $q \in \mathcal{H}[a, n]$, $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$, and suppose that*

$$\varphi(q(z), tzq'(z); \zeta, \xi) \in h(U), \tag{6}$$

for $z \in U$, $\zeta \in \partial U$, $\xi \in \overline{U}$ and $0 < t \leq \dfrac{1}{n} \leq 1$.

If $p \in Q(a)$ and $\varphi(p(z), zp'(z); \zeta, \xi)$ is univalent in U, for all $\xi \in \overline{U}$, then

$$h(z) \prec\prec \varphi(p(z), zp'(z); z, \xi) \tag{7}$$

implies

$$q(z) \prec p(z), \quad z \in U.$$

Definition 5. A strong differential superordination of the form

$$h(z) \prec\prec A(z,\xi)zp'(z) + B(z,\xi)p(z), \quad z \in U,\ \xi\overline{U}, \tag{8}$$

where h is analytic in U, and $A(z,\xi)zp'(z) + B(z,\xi)p(z)$, is univalent in U, for all $\xi \in \overline{U}$, is called first order strong linear differential superordination.
Remark 1. If $A(z,\xi) = B(z,\xi) \equiv 1$, then (8) becomes

$$h(z) \prec zp'(z) + p(z), \quad z \in U, \tag{8'}$$

a differential superordination studied by S.S. Miller and P.T. Mocanu in [3].
Remark 2. If $A(z,\xi) = 1$ and $B(z,\xi) = 0$ then (8) becomes

$$h(z) \prec zp'(z), \quad z \in U, \tag{8''}$$

a differential superordination studied by S.S. Miller and P.T. Mocanu in [3].

For the first order strong differential superordination in (8) we prove the following theorems:

Theorem 3. *Let h be convex in U, with $h(0) = a$, $q \in \mathcal{H}[a,n]$, $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ and suppose that*

$$\varphi(q(z), tzq'(z); \zeta, \xi) \in h(U), \tag{9}$$

for $z \in U$, $\zeta \in \partial U$, $\xi \in \overline{U}$ and $0 < t \leq \dfrac{1}{n} \leq 1$.

If $p \in Q(a)$ and $p(z) + \dfrac{A(z,\xi)zp'(z)}{\gamma}$, $\gamma \neq 0$, is univalent in U, for all $\xi \in \overline{U}$ and

$$h(z) \prec\prec p(z) + \frac{A(z,\xi)zp'(z)}{\gamma}, \quad z \in U,\ \xi \in \overline{U} \tag{10}$$

then

$$q(z) \prec p(z), \quad z \in U,$$

where

$$q(z)=\frac{\gamma}{z^{\gamma}}\int_0^z h(t)t^{\gamma-1}dt. \tag{11}$$

The function q is convex.

Proof. Let $\varphi:\mathbb{C}^2\times U\times\overline{U}\to\mathbb{C}$, for $r=p(z)$, $s=zp'(z)$,

$$\varphi(p(z),zp'(z);z,\xi)=r+\frac{A(z,\xi)s}{\gamma}=p(z)+\frac{A(z,\xi)zp'(z)}{\gamma},$$

then (10) becomes

$$h(z)\prec\prec\varphi(r,s;z,\xi)=\varphi(p(z),zp'(z);z,\xi)=p(z)+\frac{A(z,\xi)zp'(z)}{\gamma}. \tag{12}$$

Since the integral operator in (11) is the one in Lemma B, by applying this lemma we obtain that q is convex. From (11) we have:

$$z^{\gamma}q(z)=\gamma\int_0^z h(t)t^{\gamma-1}dt. \tag{13}$$

Differentiating (13) with respect to z, we obtain

$$q(z)+\frac{zq'(z)}{\gamma}=h(z),\quad z\in U. \tag{14}$$

Using (9) and (14), (12) becomes

$$q(z)+\frac{zq'(z)}{\gamma}=h(z)\prec\prec\varphi(p(z),zp'(z);z,\xi)=p(z)+\frac{A(z,\xi)zp'(z)}{\gamma}.$$

By applying Theorem 2 we have $q(z)\prec p(z)$, $z\in U$.

Remark 3. For $A(z,\xi)\equiv 1$, the result was obtained in [3, Theorem 6.].

Theorem 4. *Let h be starlike in U, with $h(0)=0$ $q\in\mathcal{H}[0,1]$, $\varphi:\mathbb{C}^2\times U\times\overline{U}\to\mathbb{C}$ and suppose that*

$$\varphi(q(z),tzq'(z);\zeta,\xi)\in h(U), \tag{15}$$

for $z \in U$, $\zeta \in \partial U$, $\xi \in \overline{U}$ *and* $0 < t \leq \frac{1}{n} \leq 1$.

If $p \in \mathcal{H}[0,1] \cap Q$ $(p \in Q(0))$ *and* $zp'(z)B(z,\xi)$ *is univalent in* U, *for all* $\xi \in \overline{U}$, *then*

$$h(z) \prec\prec zp'(z)B(z,\xi) \tag{16}$$

implies

$$q(z) \prec p(z), \quad z \in U,$$

where

$$q(z) = \int_0^z h(t)t^{-1}dt. \tag{17}$$

The function q *is convex.*

Proof. Differentiating (17), we obtain

$$zq'(z) = h(z), \quad z \in U.$$

Since h is starlike, from the Duality theorem of Alexander we have that q is convex.

Let $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$, $\varphi(s; z, \xi) = \varphi(zp'(z); z, \xi) = zp'(z)B(z,\xi)$. Then (16) becomes

$$h(z) \prec\prec \varphi(zp'(z); z, \xi), \quad z \in U,\ \xi \in \overline{U}, \tag{18}$$

by using (15) and applying Theorem 2 we have $q(z) \prec p(z)$, $z \in U$.

Remark 4. For $B(z,\xi) \equiv 1$, the result was obtained in [3, Theorem 9].

Example 1. Let $h(z) = z$, from Theorem 4,

$$q(z) = \int_0^z t \cdot t^{-1}dt = z.$$

If $p \in \mathcal{H}[0,1] \cap Q$ and $zp'(z)B(z,\xi)$ is univalent in U, for all $\xi \in \overline{U}$, then

$$z \prec\prec zp'(z)B(z,\xi), \quad z \in U,\ \xi \in \overline{U}$$

implies

$$z \prec p(z), \quad z \in U.$$

Example 2. Let $h(z) = z + \dfrac{z^2}{2}$, from Theorem 4,

$$q(z) = \int_0^z \left(t + \frac{t^2}{2}\right) t^{-1} dt = \int_0^z \left(1 + \frac{t}{2}\right) dt = z + \frac{z^2}{4}.$$

$$\operatorname{Re} \frac{zh'(z)}{h(z)} = \operatorname{Re} \frac{2(1+z)}{2+z} = 1 + \frac{2(1+\cos\theta)}{2\cos\theta + 5} > 0,$$

hence h is starlike in U.

If $p \in \mathcal{H}[0,1] \cap Q$ and $zp'(z)B(z,\xi)$ is univalent in U, for all $\xi \in \overline{U}$, then

$$z + \frac{z^2}{2} \prec\prec zp'(z)B(z,\xi), \quad z \in U,\ \xi \in \overline{U}$$

implies

$$z + \frac{z^2}{4} \prec p(z), \quad z \in U.$$

Theorem 5. *Let $|a| < 1$ and $r = r(a) = \dfrac{1-|a|}{2}$ and $\lambda : \overline{U} \to \mathbb{C}$ with $|\lambda(z,\xi)| \le 1$. If $p \in \mathcal{H}[ar,1] \cap Q$ and $p(z) + \lambda(z,\xi)zp'(z)$ is univalent in U, for all $\xi \in \overline{U}$, then*

$$U \subset \{p(z) + \lambda(z,\xi)zp'(z)|,\ z \in U,\ \xi \in \overline{U}\} \tag{19}$$

implies

$$U_r \subset p(U).$$

Proof. Let $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$, $\varphi(r, s; z, \xi) = r + \lambda(z, \xi)s$ where $r = p(z)$, $s = zp'(z)$, and

$$q(z) = r\frac{z+a}{1+\overline{a}z},$$

then q is univalent, $q(U) = U_r$, and (19) can be written in the form

$$U \subset \{\varphi(p(z), zp'(z); z, \xi) \mid z \in U,\ \xi \in \overline{U}\}$$

We evaluate

$$|\varphi(q(z), tzq'(z); \zeta, \xi)| = |q(z) + \lambda(z, \xi)zq'(z)|$$
$$= \left| r\frac{z+a}{1+\overline{a}z} + \lambda(z,\xi)tr\frac{(1-|a|^2)z}{(1+\overline{a}z)^2} \right| \le \left| r + tr\frac{1-|a|^2}{(1-|a|)^2} \right|$$
$$\le r\left[1 + \frac{1+|a|}{1-|a|}\right] \le r\frac{2}{1-|a|} \le 1,$$

from which we have $\varphi(q(z), tzq'(z); \zeta, \xi) \in U$.

Since $\varphi(q(z), tzq'(z); \zeta, \xi) \in U$ and from (19), by applying Theorem 1 we obtain

$$q(z) \prec p(z), \text{ i.e. } U_r \subset p(U).$$

Remark 5. For $a = 0$, and $\lambda(z, \xi) = 1$, $r = \dfrac{1}{2}$, we obtain in [3, Corollary 10.1].

Example 3. Let $a = \dfrac{1}{2} + \dfrac{1}{2}i$, $r = \dfrac{2-\sqrt{2}}{4}$.

If $p \in \left[\dfrac{a(2-\sqrt{2})}{4}, 1\right] \cap Q$ and $p(z) + \lambda(z, \xi)zp'(z)$ is univalent in U, for all $\xi \in \overline{U}$, with $|\lambda(z, \xi)| \le 1$, then

$$U \subset \{p(z) + \lambda(z, \xi)zp'(z); z, \xi \mid z \in U,\ \xi \in \overline{U}\}$$

implies

$$U_r \subset p(U).$$

References

[1] José A. Antonino and Salvador Romaguera, *Strong differential subordination to Briot-Bouquet differential equations*, Journal of Differential Equations 114(1994), 101-105.

[2] S.S. Miller and P.T. Mocanu, *Differential subordinations. Theory and applications*, Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 2000.

[3] S.S. Miller and P.T. Mocanu, *Subordinants of differential superordinations*, Complex Variables, vol.48, no.10, 815-826.

[4] Georgia Irina Oros and Gh. Oros, *Strong differential subordination* (to appear).

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