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Euler polynomials associated with p-adic q-Euler measure

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Abstract

In this paper we define two variable q-l-function. By applying Hankel's contour and Cauchy-Residue Theorem, we prove that this function interpolates generalized q-Euler numbers at negative integers. The main purpose of this paper is also to construct p-adic q-Euler measure on $\mathbb{Z}_p$ and to give applications of this measure. Furthermore, we obtain relations between p-adic q-integral, p-adic q-Euler measure and the q-Euler numbers and polynomials.

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1 Introduction, definitions and notations

In this section, we give some notations and definitions, which are used in this paper.

Let p be a fixed odd prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of the algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$, cf. ([3], [4], [5]). When we talk of q-extension, q is variously considered as an indeterminate, either a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we normally assume $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$, cf. ([6], [7], [21]).

For a fixed positive integer d with $(p, d) = 1$, set

$$\mathbb{X}_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z}, \quad \mathbb{X}_1 = \mathbb{Z}_p,$$

$$\mathbb{X}^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^N\mathbb{Z}_p = \left\{ x \in \mathbb{X} : x \equiv a(moddp^N) \right\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$, cf. ([21], [15]).

For a uniformly differentiable function f at a point $a \in \mathbb{Z}_p$ we write $f \in UD(\mathbb{Z}_p)$, if the difference quotient

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y},$$

has a limit $f'(a)$ as $(x,y) \to (a,a)$. For $f \in UD(\mathbb{Z}_p)$, an invariant p-adic q-integral was defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x,$$

where

$$[x]_q = \begin{cases} \frac{1-q^x}{1-q} & ,q \neq 1 \\ x & ,q = 1 \end{cases},$$

and

$$[x]_{-q} = \frac{1-(-q)^x}{1+q},$$

The modified p-adic q-integral on $\mathbb{Z}_p$ is defined by

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x), \tag{1}$$

where $d\mu_{-q}(x) = \lim_{q\to -q} d\mu_q(x)$ cf. ([10], [2], [7], [8], [3], [4], [9], [11], [6], [21], [16]).

The classical Euler numbers are defined by the following generating function

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad |t| < \pi,$$

From the above function, we have

$$E_0 = 1,\ E_1 = \frac{-1}{2},\ E_2 = 0,\ E_3 = \frac{1}{4}, \cdots .$$

These numbers are interpolated by the following function at the negative integers:

$$\zeta_E(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s},\ s \in \mathbb{C}. \tag{2}$$

This function interpolates Euler numbers at negative integers. For $s = -n$, $n \in \mathbb{Z}^+$, we have

$$\zeta_E(-n) = E_n,$$

cf. (see for detail [12], [22], [21], [3], [4], [5], [6], [7], [21], [1], [10], [2], [8], [11], [14], [13], [17], [18], [19], [20]).

The main motivation of this paper are summarized as follows:

In Section 2, we define two variable q-l-functions. By using Hankel's contour and Cauchy-Residue Theorem, we find explicit values of the two variable q-l-functions at negative integers.

In Section 3, we construct p-adic q-Euler measure on $\mathbb{Z}_p$. By using this measure, we prove relations between p-adic q-integral, p-adic q-Euler measure and the q-Euler numbers and polynomials. We also give some applications as well.

2 Interpolation functions of the q-Euler numbers and polynomials on $\mathbb{C}$

In this chapter, we assume that $q \in \mathbb{C}$, with $|q| < 1$.

q-extension of Euler polynomials, $E_{n,q}(x)$ are defined by

$$F_q(t,x) = \frac{2e^{tx}}{qe^t + 1} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!} \text{ cf. [14].} \tag{3}$$

By using (3), and Taylor series of e^{tx}, we have

$$\sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}\sum_{n=0}^{\infty} x^n\frac{t^n}{n!} = \sum_{n=0}^{\infty} E_{n,q}^{(h)}(x)\frac{t^n}{n!}.$$

By Cauchy product in the above, we have the following theorem:

Theorem 1. *([14]) Let $n \in \mathbb{N}$. Then we have*

$$E_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} E_{k,q}^{(h)}(x). \tag{4}$$

Theorem 2. *([14])(Distribution Relation) For d is an odd positive integer, $k \in \mathbb{N}$, we have*

$$E_{k,q}(x, q) = d^k \sum_{a=0}^{d-1} (-1)^a q^a E_{k,q^d}\left(\frac{x+a}{d}\right). \tag{5}$$

By applying Mellin transform to (3), we define Hurwitz type zeta function as follows:

$$\zeta_q(s, x) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} F_q(-t, x) dt, \tag{6}$$

(for detail see also [14]).

This function interpolates $E_{n,q}(x)$ polynomial at negative integers. By using the complex integral representation of generating function of the polynomials $E_{n,q}(x)$, we have

$$\frac{1}{\Gamma(s)} \oint_C t^{s-1} F_q(-t, x) dt = \sum_{n=0}^{\infty} \frac{(-1)^n E_{n,q}(x)}{n!} \frac{1}{\Gamma(s)} \oint_C t^{n+s-1} dt,$$

where C is Hankel's contour along the cut joining the points $z = 0$ and $z = \infty$ on the real axis, which starts from the point at ∞, encircles the origin ($z = 0$) once in the positive (counter-clockwise) direction, and returns to the point at ∞, (see for detail [23], [11], [19], [21]). By using (6) and Cauchy-Residue Theorem, we arrive at the following theorem:

Theorem 3. *Let $k \in \mathbb{N}$. Then we have*

$$\zeta_q(-k, x) = E_{k,q}(x). \tag{7}$$

Generalized q-Euler polynomials are defined by means of the following generating function [14]:

(8)

$$F_q(t,x,\chi) = \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{t(a+x)}q^a}{qe^{td}+1} = \sum_{n=0}^{\infty} E_{n,\chi,q}(x)\frac{t^n}{n!}, \quad |t+\log q| < \frac{\pi}{d}.$$

Remark 1. *From the above generating function we assume that d is an odd integer, we have*

(9)

$$\begin{aligned} F_q(t,x,\chi) &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{t(a+x)}q^a}{qe^{td}+1} \\ &= 2\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{t(a+x)}q^a\sum_{n=0}^{\infty}(-1)^n q^n e^{tdn} \\ &= 2\sum_{m=0}^{\infty}(-1)^m\chi(m)q^m e^{(m+x)t} \end{aligned}$$

By applying Mellin transform to (9), we define two variable q-l-function as follows:

(10)

$$\begin{aligned} l_q(s,\chi;x) &= \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}F_q(-t,x,\chi)dt \\ &= 2\sum_{n=0}^{\infty}\frac{(-1)^n\chi(n)q^n}{(n+x)^s}. \end{aligned}$$

Definition 1. *Let $s \in \mathbb{C}$. We define*

$$l_q(s,\chi;x) = 2\sum_{n=0}^{\infty}\frac{(-1)^n\chi(n)q^n}{(n+x)^s}.$$

Observe that if $x = 1$, then $l_q(s,\chi;x)$ reduces to

$$l_q(s,\chi;x) = 2\sum_{n=1}^{\infty}\frac{(-1)^n\chi(n)q^n}{n^s}.$$

This function interpolates q-generalized Euler numbers at negative integers. And

$$\lim_{q\to 1} l_q(s,\chi) = l(s,\chi) = 2\sum_{n=1}^{\infty} \frac{(-1)^n \chi(n)}{n^s},$$

this function interpolates generalized Euler numbers at negative integers. Substituting $\chi \equiv 1$ into the above, then the function $l(s,1)$ reduces to (2).

By using the complex integral representation of generating function in (9), we have

$$\frac{1}{\Gamma(s)} \oint_C t^{s-1} F_q(-t,x,\chi)dt = \sum_{n=0}^{\infty} \frac{(-1)^n E_{n,\chi,q}(x)}{n!} \frac{1}{\Gamma(s)} \oint_C t^{n+s-1} dt,$$

where C is Hankel's contour along the cut joining the points $z = 0$ and $z = \infty$ on the real axis, which starts from the point at ∞, encircles the origin ($z = 0$) once in the positive (counter-clockwise) direction, and returns to the point at ∞. By using (10) and Cauchy-Residue Theorem, we arrive at the following theorem:

Theorem 4. *Let $k \in \mathbb{N}$. Then we have*

$$l_q(-k,\chi;x) = E_{n,\chi,q}(x).$$

Remark 2. *Proofs of Theorem 2 and Theorem 3 were given by Ozden and Simsek. Their proofs are related to derivative operator on generating functions of the q-Euler polynomials and generalized q-Euler polynomials.*

3 p-adic q-Euler measure on $\mathbb{X}$

In this section, we assume that $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Let χ be a primitive Dirichlet character with a conductor $d(= odd) \in \mathbb{N}$.

By using (5), we define a distribution on $\mathbb{X}$. By using this distribution, we construct a measure on $\mathbb{X}$. We give relations between p-adic q-Euler measure, p-adic q-integral and q-Euler numbers and polynomials.

Let N, k and d $(= odd)$ be positive integers. We define $\mu_k^* = \mu_{k,q;E}^*$ as follows:

$$\mu_k^*(a + dp^N\mathbb{Z}_p) = (-1)^a\,(dp^N)^{k-1} q^a E_k\left(\frac{a}{dp^N}, q^{dp^N}\right). \tag{11}$$

Now we show that $\mu_k^*(a + dp^N\mathbb{Z}_p)$ is a distribution on $\mathbb{X}$ as follows:

By using (5) and (11), we obtain

$$\begin{aligned}
&\sum_{j=0}^{p-1} \mu_k^*(a + jdp^N + dp^{N+1}\mathbb{Z}_p) \\
= &\sum_{j=0}^{p-1} (-1)^{a+jdp^N}\,(dp^{N+1})^{k-1} q^{a+jdp^N} E_k\left(\frac{a+jdp^N}{dp^{N+1}}, q^{dp^{N+1}}\right) \\
= &(-1)^a\, q^a\, (dp^{N+1})^{k-1} \sum_{j=0}^{p-1} (-1)^{jdp^N}\, q^{jdp^N} E_k\left(\frac{\frac{a}{dp^N} + j}{p}, (q^{dp^N})^p\right) \\
= &(-1)^a\, q^a\, (dp^N)^{k-1} p^{k-1} \sum_{j=0}^{p-1} (-1)^j\, (q^{dp^N})^j E_k\left(\frac{\frac{a}{dp^N} + j}{p}, (q^{dp^N})^p\right) \\
= &(-1)^a\, q^a\, (dp^N)^{k-1} p^{k-1} E_k\left(\frac{a}{dp^N}, q^{dp^N}\right) \\
= &\mu_k^*(a + dp^N\mathbb{Z}_p).
\end{aligned}$$

Therefore we easily arrive at the following theorem

Theorem 5. *Let N, k and d $(= odd)$ be positive integers, then*

$$\mu_k^*(a + dp^N\mathbb{Z}_p) = (-1)^a\,(dp^N)^{k-1} q^a E_k\left(\frac{a}{dp^N}, q^{dp^N}\right)$$

is a distribution on $\mathbb{X}$.

Substituting $f(x) = q^x e^{tx}$ into (1), we obtain (3) cf. [14]. By using (3), we have

$$\frac{2e^{tx}}{qe^t+1} = \sum_{n=0}^{\infty}(-1)^n q^n e^{t(n+x)}. \tag{12}$$

From the above series and Theorem 5, we arrive at the following theorem:

Theorem 6. *If $q \in \mathbb{Z}_p$ with $|1-q|_p \leq 1$, then μ_k^* is a measure on $\mathbb{X}$.*

Proof. From Theorem 5, (5) and (12) we easily arrive at the desired result.

Theorem 7. *For any positive integer k, we have*

$$\int_{\mathbb{Z}_p} d\mu_k^*(x) = E_k(q).$$

Proof. By Theorem 6 we have

$$\begin{aligned}\int_{\mathbb{Z}_p} d\mu_k^*(x) &= \lim_{N\to\infty}\sum_{x=0}^{dp^N-1}\mu_k^*(x+dp^N\mathbb{Z}_p)\\ &= \lim_{N\to\infty}\sum_{a=0}^{d-1}\sum_{j=0}^{p^N-1}\mu_k^*(a+jd+dp^N\mathbb{Z}_p).\end{aligned}$$

By using Theorem 5, we get

$$\begin{aligned}&= \lim_{N\to\infty}\sum_{a=0}^{d-1}\sum_{j=0}^{p^N-1}(-1)^{a+jd}(dp^N)^{k-1}q^{a+jd}E_k\left(\frac{a+jd}{dp^N}, q^{dp^N}\right)\\ &= \sum_{a=0}^{d-1}(-1)^a q^a d^{k-1}\lim_{N\to\infty}(p^N)^{k-1}\sum_{j=0}^{p^N-1}(-1)^j(q^j)^d E_k\left(\frac{\frac{a}{d}+j}{p^N}, (q^d)^{p^N}\right)\\ &= \sum_{a=0}^{d-1}(-1)^a q^a d^{k-1}E_k\left(\frac{a}{d}, q^d\right)\\ &= E_k(q).\end{aligned}$$

Thus we complete the proof.

Theorem 8. *Let χ be the Dirichlet's character with an odd conductor $d \in \mathbb{N}$. Then we have*

$$\int_{\mathbb{X}} \chi(x) d\mu^*_k(x) = E_{k,\chi}(q).$$

Proof.

$$\begin{aligned}
\int_{\mathbb{X}} \chi(x) d\mu^*_k(x) &= \lim_{N\to\infty} \sum_{x=0}^{dp^N-1} \chi(x)\mu^*_k(x + dp^N\mathbb{Z}_p) \\
&= \lim_{N\to\infty} \sum_{a=0}^{d-1} \sum_{j=0}^{p^N-1} \chi(a + jd)\mu^*_k(a + jd + dp^N\mathbb{Z}_p) \\
&= \lim_{N\to\infty} \sum_{a=0}^{d-1} \chi(a) \sum_{j=0}^{p^N-1} (-1)^{a+jd} q^{a+jd} (dp^N)^{k-1} E_k(\frac{a + jd}{dp^N}, q^{dp^N}) \\
&= d^{k-1} \sum_{a=0}^{d-1} (-1)^a q^a \chi(a) \\
&\quad \times \lim_{N\to\infty} (p^N)^{k-1} \sum_{j=0}^{p^N-1} (-1)^j (q^d)^j E_k(\frac{\frac{a}{d} + j}{p^N}, (q^d)^{p^N}) \\
&= d^{k-1} \sum_{a=0}^{d-1} (-1)^a q^a \chi(a) E_k(\frac{a}{d}, q^d) \\
&= E_k,\chi(q)
\end{aligned}$$

Remark 3. *By using μ^*_k on $\mathbb{X}^*$, and $\int_{\mathbb{X}^*} f(x)\chi(x) d\mu^*_k(x)$, we may have many applications related to p-adic l-function and q-generalized Euler numbers.*

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References

[1] L-C. Jang, *On a q-analogue of the p-adic generalized twisted L-functions and p-adic q-integrals*, J. Korean Math. Soc. 44(1) (2007), 1-10.

[2] T. Kim, *q-Volkenborn integation,* Russ. J. Math. Phys. 9 (2002), 288-299.

[3] T. Kim, *The modified q-Euler numbers and polynomials*, ArXive:math.NT/0702523.

[4] T. Kim, *On a q-analogue of the p-adic log gamma functions*, J. Number Theory 16 (1999), 320-329.

[5] T. Kim, *On the q-extension of Euler and Genocchi numbers*, J. Math. Anal. Appl. 326 (2007), 1458-1465.

[6] T. Kim, *Sums of powers of consecutive q-integers*, Advan. Stud. Contemp. Math. 9 (2004), 15-18.

[7] T. Kim, *An invariant p-adic q-integral on* $\mathbb{Z}_p$, Appl. Math. Letters, In Press, Corrected Proof, Available online 20 February 2007.

[8] T. Kim, *On the analogs of Euler numbers and polynomials associated with p-adic q-integral on* Z_p *at* $q = -1$, J. Math. Anal. Appl. 331 (2007), 779-792.

[9] T. Kim, *q-Euler numbers and polynomials associated with p-adic q-integrals*, J. Nonlinear Math. Phys. 14(1) (2007), 15-27.

[10] T. Kim, *A new approach to q-zeta function*, J. Comput. Analy. Appl. 9 (2007), 395-400.

[11] T. Kim and S.-H. Rim, *A new Changhee q-Euler numbers and polynomials associated with p-adic q-integral*, Computers & Math. Appl. 54(4) (2007), 484-489.

[12] Q.-M. Luo and H. M. Srivastava, *Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials*, Comput. Math. Appl. 10 (2005), 631-642.

[13] H. Ozden, Y. Simsek, S-H. Rim and I. N. Cangul, *A note on p-adic q-Euler measure*, Advan. Stud. Contemp. Math., 14(2) (2007), 233-239.

[14] H. Ozden and Y. Simsek, *A new extension of q-Euler numbers and polynomials related to their interpolation functions*, preprint.

[15] S-H. Rim, Y. Simsek, V. Kurt and T. Kim, *On p-adic twisted Euler* (h, q)*-l-function,* ArXive:math.NT /0702310.

[16] S-H. Rim, T. Kim, *A note on q-Euler numbers associated with the basic q-zeta function*, Appl. Math. Letters 20(4) (2007), 366-369.

[17] Y. Simsek, *On twisted generalized Euler numbers*, Bull. Korean Math. Soc. 41(2) (2004), 299-306.

[18] Y. Simsek, *q-analogue of twisted l-series and q-twisted Euler numbers*, J. Number Theory 110(2) (2005), 267–278.

[19] Y. Simsek, *Twisted (h, q)-Bernoulli numbers and polynomials related to twisted (h, q)-zeta function and L-function*, J. Math. Anal. Appl. 324 (2006), 790-804.

[20] Y. Simsek, *q-Hardy-Berndt type sums associated with q-Genocchi type zeta and l-functions*, submitted.

[21] H. M. Srivastava, T. Kim, Y. Simsek, *q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series*, Russian J. Math. Phys. 12 (2005), 241-268.

[22] H. M. Srivastava and J. Choi, Series Associated with the Zeta and Related Functions, *Kluwer Acedemic Publishers, Dordrecht, Boston and London*, 2001.

[23] E. T. Wittaker and G. N. Watson, A Course of Modern Analysis, *4th. Edition, Cambridge University Press, Cambridge*, 1962.

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