Generalized Hypergeometric Functions and Associated Families of k-Uniformly Convex and k-Starlike Functions

H. M. Srivastava

Abstract

In this lecture, we aim at presenting a certain linear operator which is defined by means of the Hadamard product (or convolution) with a generalized hypergeometric function and then investigating its various mapping as well as inclusion properties involving such subclasses of analytic and univalent functions as (for example) k-uniformly convex functions and k-starlike functions. Relevant connections of the definitions and results presented in this lecture with those in several earlier and recent works on the subject are also pointed out.

k-uniformly convex functions, k-starlike functions, parabolic starlike functions, mapping and inclusion properties, coefficient inequalities, fractional calculus operators, Gauss summation theorem.

1. Introduction, Definitions and Preliminaries

As usual, we denote by $\mathcal{A}$ the class of functions f *normalized* by

$$(1) \qquad f(z) = z + \sum_{n=2}^{\infty} a_n \, z^n,$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We also denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also *univalent* in $\mathbb{U}$. Furthermore, we denote by k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ two interesting subclasses of $\mathcal{S}$ consisting, respectively, of functions which are *k-uniformly convex* and *k-starlike* in $\mathbb{U}$. We thus have

$$(2)$$
$$k\text{-}\mathcal{UCV} := \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| \qquad (z \in \mathbb{U}; \ 0 \leqq k < \infty) \right\}$$

and

$$(3)$$
$$k\text{-}\mathcal{ST} := \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (z \in \mathbb{U}; \ 0 \leqq k < \infty) \right\}.$$

The class k-$\mathcal{UCV}$ was introduced by Kanas and Wiśniowska [12], where its geometric definition and connections with the conic domains were considered. The class k-$\mathcal{ST}$ was investigated in [13]; in fact, it is related to the

class k-$\mathcal{UCV}$ by means of the well-known Alexander equivalence between the usual classes of *convex* and *starlike* functions (see also the work of Kanas and Srivastava [11] for further developments involving each of the classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$). In particular, when $k = 1$, we obtain

$$(4) \qquad 1\text{-}\mathcal{UCV} \equiv \mathcal{UCV} \qquad \text{and} \qquad 1\text{-}\mathcal{ST} \equiv \mathcal{SP},$$

where $\mathcal{UCV}$ and $\mathcal{SP}$ are the familiar classes of uniformly convex functions and parabolic starlike functions in $\mathbb{U}$, respectively (see, for details, Goodman ([9] and [10]), Ma and Minda [14], and Rønning [22]). In fact, by making use of a certain fractional calculus operator, Srivastava and Mishra [27] presented a systematic *and* unified study of the classes $\mathcal{UCV}$ and $\mathcal{SP}$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(A, B)$ if it satisfies the following inequality:

$$(5) \qquad \left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1$$

$$(z \in \mathbb{U};\ \tau \in \mathbb{C} \setminus \{0\};\ -1 \leqq B < A \leqq 1).$$

The class $\mathcal{R}^{\tau}(A, B)$ was introduced earlier by Dixit and Pal [2]. Two of the many interesting subclasses of the class $\mathcal{R}^{\tau}(A, B)$ are worthy of mention here. First of all, by setting

$$\tau = e^{-i\eta} \cos\eta \ \left(-\frac{\pi}{2} < \eta < \frac{\pi}{2} \right), \quad A = 1 - 2\beta \ \ (0 \leqq \beta < 1), \quad \text{and} \quad B = -1,$$

the class $\mathcal{R}^{\tau}(A, B)$ reduces essentially to the class $\mathcal{R}_{\eta}(\beta)$ studied recently by Ponnusamy and Rønning [18], where

$$\mathcal{R}_{\eta}(\beta) := \left\{ f \in \mathcal{A} : \mathfrak{R}\left(e^{i\eta}\big(f'(z) - \beta\big) \right) > 0 \ \left(z \in \mathbb{U};\ -\frac{\pi}{2} < \eta < \frac{\pi}{2};\ 0 \leqq \beta < 1 \right) \right\}.$$

Secondly, if we put

$$\tau = 1, \quad A = \beta, \quad \text{and} \quad B = -\beta \quad (0 < \beta \leqq 1),$$

we obtain the class of functions $f \in \mathcal{A}$ satisfying the following inequality:

$$(6) \qquad \left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \beta \qquad (z \in \mathbb{U}; \ 0 < \beta \leqq 1),$$

which was studied by (among others) Padmanabhan [16] and Caplinger and Causey [1].

Next we introduce the classes $\mathcal{S}_\lambda^*$ and $\mathcal{C}_\lambda$ by (*cf.*, *e.g.*, [18] for the class $\mathcal{S}_\lambda^*$)

$$(7) \qquad \mathcal{S}_\lambda^* := \left\{ f \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < \lambda \quad (z \in \mathbb{U}; \ \lambda > 0) \right\}$$

and

$$(8) \qquad \mathcal{C}_\lambda := \left\{ f \in \mathcal{A} : \left| \frac{zf''(z)}{f'(z)} \right| < \lambda \quad (z \in \mathbb{U}; \ \lambda > 0) \right\},$$

so that, obviously,

$$(9) \qquad f(z) \in \mathcal{C}_\lambda \Longleftrightarrow zf'(z) \in \mathcal{S}_\lambda^* \qquad (\lambda > 0),$$

which is analogous to the aforementioned Alexander equivalence (see, for details, the monograph by Duren [3]).

Finally, we recall a sufficiently adequate *special* case of a convolution operator which was introduced earlier by Dziok and Srivastava [4] by means of the Hadamard product (or convolution) involving generalized hypergeometric functions. Indeed, by employing the Pochhammer symbol (or the *shifted*

factorial, since $(1)_n = n!$) $(\lambda)_n$ given, in terms of the Gamma functions, by

(10)

$$(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N} := \{1, 2, 3, \ldots\}), \end{cases}$$

a generalized hypergeometric function $_pF_q$ with p numerator parameters $\alpha_j \in \mathbb{C}$ $(j = 1, \ldots, p)$ and q denominator parameters

$$\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \qquad \left(\mathbb{Z}_0^- := \{0, -1, -2, \ldots\}; \ j = 1, \ldots, q\right)$$

is defined by (*cf.*, *e.g.*, [19, p. 19 *et seq.*])

(11)
$$\begin{aligned} _pF_q(z) &= {_pF_q}(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) \\ &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \end{aligned}$$

$$(p, q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \quad p < q + 1 \quad \text{and} \quad z \in \mathbb{C};$$

$$p = q + 1 \quad \text{and} \quad z \in \mathbb{U}; \ p = q + 1, \ z \in \partial\mathbb{U}, \quad \text{and} \quad \mathfrak{R}(\omega) > 0),$$

where an empty product is to be interpreted as 1 and

(12)
$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j.$$

We thus obtain (see [4, p. 3], [5] and [6]; see also the more recent works [17] and [30] dealing extensively with the Dziok-Srivastava operator)

(13)
$$\left(I_{\beta_1, \ldots, \beta_q}^{\alpha_1, \ldots, \alpha_p} f\right)(z) := z \ _pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) * f(z)$$

$$(f \in \mathcal{A}; \; p \leqq q + 1; \; z \in \mathbb{U}),$$

so that, for a function f of the form (1), we have

(14)
$$\left(I_{\beta_1,\ldots,\beta_q}^{\alpha_1,\ldots,\alpha_p} f \right)(z) = z + \sum_{n=2}^{\infty} \Gamma_n \, a_n \, z^n,$$

where, for convenience,

(15) $$\Gamma_n := \frac{(\alpha_1)_{n-1} \cdots (\alpha_p)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_q)_{n-1}} \cdot \frac{1}{(n-1)!} \qquad (n \in \mathbb{N} \setminus \{1\}).$$

Just as it was observed by Dziok and Srivastava [4, pp. 3 and 4], the convolution operator defined by (13) includes, as its *further* special cases, various other linear operators which were considered in many earlier works. In particular, for $p = 2$ and $q = 1$, we obtain the linear operator $\mathcal{F}(\alpha, \beta, \gamma)$ defined by

$$\left(\mathcal{F}(\alpha, \beta, \gamma) f \right)(z) := z \; {}_2F_1(\alpha, \beta, \gamma; z) * f(z)$$

(16)
$$= \left(I_\gamma^{\alpha,\beta} \, f \right)(z),$$

which was investigated by Hohlov [10].

It may be of interest to remark here that many univalence, starlikeness, and convexity properties of the hypergeometric functions:

$$z \; {}_2F_1(\alpha, \beta; \gamma; z)$$

and

$$z \; {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) \qquad (p \leqq q + 1)$$

were investigated in a number of earlier works (*cf.*, *e.g.*, [15], [18], [19], and [23]; see also [28] and [29]).

Our main objective in this lecture is to demonstrate the usefulness of the linear operator defined by (13) in order to establish a number of connections between the classes k-$\mathcal{UCV}$, k-$\mathcal{ST}$, $\mathcal{R}^\tau(A, B)$, and various other subclasses of $\mathcal{A}$ including (for example) the classes $\mathcal{S}_\lambda^*$ and $\mathcal{C}_\lambda$ defined by (7) and (8), respectively. The various results presented here are based essentially upon the recent investigation by Gangadharan *et al.* [7]. For several further closely-related results dealing with many of the above-defined as well as other interesting function classes, we may cite the works by (for example) Ramachandran *et al.* ([20] and [21]) and Srivastava *et al.* ([25] and [28]).

Each of the following lemmas will be required in the investigation presented here.

Lemma 1 (Dixit and Pal [2]). *If* $f \in \mathcal{R}^\tau(A, B)$ *is of the form* (1), *then*

$$(17) \qquad |a_n| \leqq (A - B)\frac{|\tau|}{n} \qquad (n \in \mathbb{N} \setminus \{1\}).$$

The estimate in (17) *is sharp for the function*:

$$(18) \quad f(z) = \int_0^1 \left(1 + (A - B)\frac{\tau t^{n-1}}{1 + Bt^{n-1}}\right) dt \qquad (z \in \mathbb{U};\ n \in \mathbb{N} \setminus \{1\}).$$

Lemma 2 (Dixit and Pal [2]). *Let* $f \in \mathcal{A}$ *be of the form* (1). *If*

$$(19) \qquad \sum_{n=2}^{\infty} (1 + |B|)\, n\, |a_n| \leqq (A - B)\, |\tau|$$

$$(-1 \leqq B < A \leqq 1;\ \tau \in \mathbb{C} \setminus \{0\}),$$

then $f \in \mathcal{R}^\tau(A, B)$. *The result is sharp for the function*:

$$(20) \qquad f(z) = z + \frac{(A - B)\,\tau}{(1 + |B|)\, n}\, z^n \qquad (z \in \mathbb{U};\ n \in \mathbb{N} \setminus \{1\}).$$

Lemma 3 (Kanas and Wiśniowska [12]). *Let $f \in \mathcal{A}$ be of the form* (1). *If, for some k $(0 \leqq k < \infty)$, the following inequality*:

$$(21) \qquad \sum_{n=2}^{\infty} n\,(n-1)\,|a_n| \leqq \frac{1}{k+2}$$

holds true, then $f \in k\text{-}\mathcal{UCV}$. The number $1/(k+2)$ cannot be increased.

Lemma 4 (Kanas and Wiśniowska [13]). *Let $f \in \mathcal{A}$ be of the form* (1). *If, for some k $(0 \leqq k < \infty)$, the following inequality*:

$$(22) \qquad \sum_{n=2}^{\infty} \{n + (n-1)\,k\}\,|a_n| < 1$$

holds true, then $f \in k\text{-}\mathcal{ST}$.

2. Mapping and Inclusion Properties Involving the Function Classes k - $\mathcal{UCV}$ and k - $\mathcal{ST}$

In this section, we first state and prove a mapping and inclusion property of the convolution operator defined by (13) involving the function class k-$\mathcal{UCV}$.

Theorem 1. *Suppose that*

$$\alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \ldots, p), \qquad \Re(\beta_j) > 0 \quad (j = 1, \ldots, q),$$

and (in the case when $p = q + 1$)

$$\Re\left(\sum_{j=1}^{q} \beta_j\right) > 1 + \sum_{j=1}^{p} |\alpha_j|.$$

If $f \in \mathcal{R}^{\tau}(A, B)$ and, for some k $(0 \leqq k < \infty)$, the following hypergeometric inequality:

$$
\begin{aligned}
&{}_pF_q\left(|\alpha_1| + 1, \ldots, |\alpha_p| + 1;\ \Re(\beta_1) + 1, \ldots, \Re(\beta_q) + 1; 1\right) \\
(23) \qquad &\leqq \frac{\Re(\beta_1) \cdots \Re(\beta_q)}{(k+2)(A-B)\,|\tau| \cdot |\alpha_1 \cdots \alpha_p|} \qquad (0 \leqq k < \infty)
\end{aligned}
$$

holds true, then

$$I_{\beta_1,\ldots,\beta_q}^{\alpha_1,\ldots,\alpha_p} f \in k\text{-}\mathcal{UCV}.$$

Proof. At the outset, under the first two parametric constraints stated in Theorem 1, it is easily seen from (15) and (10) that

$$
\begin{aligned}
|\Gamma_n| &= \frac{\left|(\alpha_1)_{n-1}\right| \cdots \left|(\alpha_p)_{n-1}\right|}{\left|(\beta_1)_{n-1}\right| \cdots \left|(\beta_q)_{n-1}\right|} \cdot \frac{1}{(n-1)!} \\
&\leqq \frac{(|\alpha_1|)_{n-1} \cdots (|\alpha_p|)_{n-1}}{(\Re(\beta_1))_{n-1} \cdots (\Re(\beta_q))_{n-1}} \cdot \frac{1}{(n-1)!} \\
&= \frac{1}{n-1} \cdot \frac{|\alpha_1 \cdots \alpha_p|}{\Re(\beta_1) \cdots \Re(\beta_q)}
\end{aligned}
$$

(24)
$$
\cdot \frac{(|\alpha_1|+1)_{n-2} \cdots (|\alpha_p|+1)_{n-2}}{(\Re(\beta_1)+1)_{n-2} \cdots (\Re(\beta_q)+1)_{n-2}} \cdot \frac{1}{(n-2)!} \qquad (n \in \mathbb{N} \setminus \{1\}).
$$

Thus, for $f \in \mathcal{R}^\tau(A, B)$ of the form (1), by applying Lemma 1 in conjunction with (24), we have

$$
\begin{aligned}
\sum_{n=2}^{\infty} & n(n-1) |\Gamma_n| \cdot |a_n| \\
&\leqq \frac{(A-B)|\tau| \cdot |\alpha_1 \cdots \alpha_p|}{\Re(\beta_1) \cdots \Re(\beta_q)} \\
&\qquad \cdot \sum_{n=2}^{\infty} \frac{(|\alpha_1|+1)_{n-2} \cdots (|\alpha_p|+1)_{n-2}}{(\Re(\beta_1)+1)_{n-2} \cdots (\Re(\beta_q)+1)_{n-2}} \cdot \frac{1}{(n-2)!} \\
&= \frac{(A-B)|\tau| \cdot |\alpha_1 \cdots \alpha_p|}{\Re(\beta_1) \cdots \Re(\beta_q)}
\end{aligned}
$$

(25)
$$
\cdot {}_pF_q\left(|\alpha_1|+1,\ldots,|\alpha_p|+1;\ \Re(\beta_1)+1,\ldots,\Re(\beta_q)+1;1\right),
$$

where the convergence of the ${}_pF_q(1)$ series is guaranteed (*when* $p = q+1$) by the *third* parametric constraint stated in Theorem 1 by analogy with the inequality (12).

Finally, if we make use of the hypothesis (23) in (25), we find that

$$(26) \qquad \sum_{n=2}^{\infty} n\,(n-1)\,|\Gamma_n| \cdot |a_n| \leqq \frac{1}{k+2} \qquad (0 \leqq k < \infty),$$

which, in view of (14) *and* Lemma 3, immediately proves the mapping and inclusion property asserted by Theorem 1.

Theorem 1 can be applied to deduce the corresponding mapping and inclusion properties, involving the class k-$\mathcal{UCV}$, for all those linear operators (listed by Dziok and Srivastava [4, pp. 3 and 4]), which happen to be *further* special cases of the convolution operator defined by (13). In particular, for the Hohlov operator $\mathcal{F}(\alpha, \beta, \gamma)$ defined by (16), by appealing to the *Gauss summation theorem* [26, p. 9, Equation 1.2 (20)]:

$$(27) \qquad \qquad {}_2F_1\,(a, b; c; 1) = \frac{\Gamma\,(c)\,\Gamma\,(c - a - b)}{\Gamma\,(c - a)\,\Gamma\,(c - b)}$$

$$\left(\Re\,(c - a - b) > 0;\; c \in \mathbb{C} \setminus \mathbb{Z}_0^-\right),$$

Theorem 1 yields

Corollary 1. *Let γ be a real number such that*

$$\gamma > |\alpha| + |\beta| + 1 \qquad (\alpha, \beta \in \mathbb{C} \setminus \{0\}).$$

If $f \in \mathcal{R}^\tau\,(A, B)$ and, for some k $(0 \leqq k < \infty)$, the following inequality:

(28)
$$\frac{\Gamma\,(\gamma)\,\Gamma\,(\gamma - |\alpha| - |\beta| - 1)}{\Gamma\,(\gamma - |\alpha|)\,\Gamma\,(\gamma - |\beta|)} \leqq \frac{1}{(k+2)\,(A - B)\,|\tau| \cdot |\alpha\beta|} \qquad (0 \leqq k < \infty)$$

holds true, then

$$\mathcal{F}\,(\alpha, \beta, \gamma)\,f \in k\text{-}\mathcal{UCV}.$$

In a similar manner, by applying Lemma 1 *and* Lemma 4 (instead of Lemma 3), we can prove the following mapping and inclusion property, involving the class $k\text{-}\mathcal{ST}$, for the convolution operator defined by (13).

Theorem 2. *Suppose that*

$$\alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \ldots, p), \qquad \mathfrak{R}(\beta_j) > 0 \quad (j = 1, \ldots, q),$$

and (in the case when $p = q + 1$)

$$\mathfrak{R}\left(\sum_{j=1}^{q} \beta_j\right) > \sum_{j=1}^{p} |\alpha_j|.$$

If $f \in \mathcal{R}^\tau(A, B)$ and, for some k $(0 \leqq k < \infty)$, the following hypergeometric inequality:

$$(k+1)\,_pF_q\left(|\alpha_1|, \ldots, |\alpha_p|; \mathfrak{R}(\beta_1), \ldots, \mathfrak{R}(\beta_q); 1\right)$$

$$- k\,_{p+1}F_{q+1}\left(|\alpha_1|, \ldots, |\alpha_p|, 1; \mathfrak{R}(\beta_1), \ldots, \mathfrak{R}(\beta_q), 2; 1\right)$$

(29) $$< 1 + 2k + \frac{1}{(A-B)\,|\tau|} \qquad (0 \leqq k < \infty)$$

holds true, then

$$I_{\beta_1,\ldots,\beta_q}^{\alpha_1,\ldots,\alpha_p} f \in k\text{-}\mathcal{ST}.$$

For $p = 2$ and $q = 1$, Theorem 2 readily yields

Corollary 2. *Let γ be a real number such that*

$$\gamma > |\alpha| + |\beta| \qquad (\alpha, \beta \in \mathbb{C} \setminus \{0\}).$$

If $f \in \mathcal{R}^\tau(A, B)$ and, for some k $(0 \leqq k < \infty)$, the following hypergeometric

inequality:

$$k \ _3F_2 \left(|\alpha| , |\beta| , 1; \gamma, 2; 1 \right)$$

(30)
$$> (k+1) \frac{\Gamma\left(\gamma\right)\Gamma\left(\gamma - |\alpha| - |\beta|\right)}{\Gamma\left(\gamma - |\alpha|\right)\Gamma\left(\gamma - |\beta|\right)} - 2k - 1 - \frac{1}{(A-B)\,|\tau|} \quad (0 \leqq k < \infty)$$

holds true, then

$$\mathcal{F}\left(\alpha, \beta, \gamma\right) f \in k\text{-}\mathcal{ST}.$$

Next, for a function f of the form (1) and belonging to the class k-$\mathcal{UCV}$, the following coefficient inequalities hold true (*cf.* [12]):

(31)
$$|a_n| \leqq \frac{(P_1)_{n-1}}{n!} \qquad (n \in \mathbb{N} \setminus \{1\}),$$

where $P_1 = P_1\left(k\right)$ is the coefficient of z in the function:

(32)
$$p_k\left(z\right) = 1 + \sum_{n=1}^{\infty} P_n\left(k\right) z^n,$$

which is the extremal function for the class $\mathcal{P}\left(p_k\right)$ related to the class k-$\mathcal{UCV}$ by the range of the following expression:

$$1 + \frac{z f''\left(z\right)}{f'\left(z\right)} \qquad (z \in \mathbb{U}).$$

Similarly, if f of the form (1) belongs to the class k-$\mathcal{ST}$, then (*cf.* [13])

(33)
$$|a_n| \leqq \frac{(P_1)_{n-1}}{(n-1)!} \qquad (n \in \mathbb{N} \setminus \{1\}),$$

where $P_1 = P_1\left(k\right)$ is given, as above, by (32).

Making use of the coefficient inequalities (31) and (33), in place of the coefficient inequality (17) asserted by Lemma 1, we can establish each of

the following results (Theorem 3 and Theorem 4 below) by appealing appropriately to Lemma 3 and Lemma 4, respectively.

Theorem 3. *Suppose that*

$$\alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \ldots, p), \qquad \Re(\beta_j) > 0 \quad (j = 1, \ldots, q),$$

and (in the case when $p = q + 1$)

$$\Re\left(\sum_{j=1}^{q} \beta_j\right) > P_1 + \sum_{j=1}^{p} |\alpha_j|,$$

where $P_1 = P_1(k)$ is given, as before, by (32). If, for some k $(0 \leqq k < \infty)$, $f \in k\text{-}\mathcal{UCV}$ and the following hypergeometric inequality:

$$_{p+1}F_{q+1}\left(|\alpha_1| + 1, \ldots, |\alpha_p| + 1, P_1 + 1; \Re(\beta_1) + 1, \ldots, \Re(\beta_q) + 1, 2; 1\right)$$

(34)

$$\leqq \frac{\Re(\beta_1) \cdots \Re(\beta_q)}{(k+2)|\alpha_1 \cdots \alpha_p| P_1} \qquad (0 \leqq k < \infty)$$

holds true, then

$$I_{\beta_1,\ldots,\beta_q}^{\alpha_1,\ldots,\alpha_p} f \in k\text{-}\mathcal{UCV}.$$

Theorem 4. *Suppose that*

$$\alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \ldots, p), \qquad \Re(\beta_j) > 0 \quad (j = 1, \ldots, q),$$

and (in the case when $p = q + 1$)

$$\Re\left(\sum_{j=1}^{q} \beta_j\right) > P_1 + \sum_{j=1}^{p} |\alpha_j|,$$

where $P_1 = P_1(k)$ is given, as before, by (32). If, for some k $(0 \leqq k < \infty)$, $f \in k\text{-}\mathcal{ST}$ and the following hypergeometric inequality:

$$\frac{(k+1)\left|\alpha_1 \cdots \alpha_p\right| P_1}{\Re(\beta_1) \cdots \Re(\beta_q)} \; {}_{p+1}F_{q+1}\left(|\alpha_1|+1, \ldots, |\alpha_p|+1, P_1+1; \Re(\beta_1)+1, \ldots, \Re(\beta_q)+1, 2; 1\right)$$

(35)

$$+ \; {}_{p+1}F_{q+1}\left(|\alpha_1|, \ldots, |\alpha_p|, P_1; \Re(\beta_1), \ldots, \Re(\beta_q), 1; 1\right) < 2 \qquad (0 \leqq k < \infty)$$

holds true, then

$$I^{\alpha_1, \ldots, \alpha_p}_{\beta_1, \ldots, \beta_q} f \in k\text{-}\mathcal{ST}.$$

The following (seemingly interesting) variants of Theorem 3 and Theorem 4 can also be proven similarly, and we omit the details involved.

Theorem 5. *Suppose that*

$$\alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \ldots, p), \qquad \Re(\beta_j) > 0 \quad (j = 1, \ldots, q),$$

and (in the case when $p = q + 1$)

$$\Re\left(\sum_{j=1}^{q} \beta_j\right) > P_1 - 1 + \sum_{j=1}^{p} |\alpha_j|,$$

where $P_1 = P_1(k)$ is given, as before, by (32). If, for some k $(0 \leqq k < \infty)$, $f \in k\text{-}\mathcal{UCV}$ and the following hypergeometric inequality:

$$\frac{(k+1)\left|\alpha_1 \cdots \alpha_p\right| P_1}{2\Re(\beta_1) \cdots \Re(\beta_q)} \; {}_{p+1}F_{q+1}\left(|\alpha_1|+1, \ldots, |\alpha_p|+1, P_1+1; \Re(\beta_1)+1, \ldots, \Re(\beta_q)+1, 3; 1\right)$$

(36)

$$+ \; {}_{p+1}F_{q+1}\left(|\alpha_1|, \ldots, |\alpha_p|, P_1; \Re(\beta_1), \ldots, \Re(\beta_q), 2; 1\right) < 2 \qquad (0 \leqq k < \infty)$$

holds true, then

$$I^{\alpha_1, \ldots, \alpha_p}_{\beta_1, \ldots, \beta_q} f \in k\text{-}\mathcal{ST}.$$

Theorem 6. *Suppose that*

$$\alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \ldots, p), \qquad \Re(\beta_j) > 0 \quad (j = 1, \ldots, q),$$

and (in the case when $p = q + 1$)

$$\Re\left(\sum_{j=1}^{q} \beta_j\right) > P_1 + 1 + \sum_{j=1}^{p} |\alpha_j|,$$

where $P_1 = P_1(k)$ is given, as before, by (32). If, for some k $(0 \leqq k < \infty)$, $f \in k\text{-}\mathcal{ST}$ and the following hypergeometric inequality:

$$_{p+2}F_{q+2}\left(|\alpha_1| + 1, \ldots, |\alpha_p| + 1, P_1 + 1, 3; \Re(\beta_1) + 1, \ldots, \Re(\beta_q) + 1, 2, 2; 1\right)$$

$$(37) \qquad \leqq \frac{\Re(\beta_1) \cdots \Re(\beta_q)}{2(k+2)|\alpha_1 \cdots \alpha_p| P_1} \qquad (0 \leqq k < \infty)$$

holds true, then

$$I_{\beta_1, \ldots, \beta_q}^{\alpha_1, \ldots, \alpha_p} f \in k\text{-}\mathcal{UCV}.$$

In its special case when $p = 2$ and $q = 1$, Theorem 3 reduces at once to the following known result:

Corollary 3 (Kanas and Srivastava [11, p. 128, Theorem 2.5]). *Let γ be a real number such that*

$$\gamma = |\alpha| + |\beta| + P_1 \qquad (\alpha, \beta \in \mathbb{C} \setminus \{0\}),$$

where $P_1 = P_1(k)$ is given, as before, by (32). If, for some k $(0 \leqq k < \infty)$, $f \in k\text{-}\mathcal{UCV}$ and the following hypergeometric inequality:

(38)

$$_3F_2\left(|\alpha| + 1, |\beta| + 1, P_1 + 1; \gamma + 1, 2; 1\right) \leqq \frac{\gamma}{(k+2)|\alpha\beta| P_1} \qquad (0 \leqq k < \infty)$$

holds true, then

$$\mathcal{F}(\alpha, \beta, \gamma) f \in k\text{-}\mathcal{UCV}.$$

For $p = 2$ and $q = 1$, Theorem 4 immediately yields the following *corrected* version of another known result:

Corollary 4 (*cf.* Kanas and Srivastava [11, p. 130, Theorem 3.5]). *Let γ be a real number such that*

$$\gamma > |\alpha| + |\beta| + P_1 \qquad (\alpha, \beta \in \mathbb{C} \setminus \{0\}),$$

where $P_1 = P_1(k)$ is given, as before, by (32). If, for some k $(0 \leqq k < \infty)$, $f \in k\text{-}\mathcal{ST}$ and the following hypergeometric inequality:

$$(39) \qquad \frac{(k+1)|\alpha\beta| P_1}{\gamma} {}_3F_2(|\alpha| + 1, |\beta| + 1, P_1 + 1; \gamma + 1, 2; 1)$$

$$+ {}_3F_2(|\alpha|, |\beta|, P_1; \gamma, 1; 1) < 2 \qquad (0 \leqq k < \infty)$$

holds true, then

$$\mathcal{F}(\alpha, \beta, \gamma) f \in k\text{-}\mathcal{ST}.$$

Similar consequences of Theorem 5 and Theorem 6 would lead us, respectively, to Corollary 5 and Corollary 6 below.

Corollary 5. *Let γ be a real number such that*

$$\gamma > P_1 - 1 + |\alpha| + |\beta| \qquad (\alpha, \beta \in \mathbb{C} \setminus \{0\}),$$

where $P_1 = P_1(k)$ is given, as before, by (32). If, for some k $(0 \leqq k < \infty)$, $f \in k\text{-}\mathcal{UCV}$ and the following hypergeometric inequality:

$$\frac{(k+1)|\alpha\beta| P_1}{2\gamma} {}_3F_2(|\alpha| + 1, |\beta| + 1, P_1 + 1; \gamma + 1, 3; 1)$$

$$(40) \qquad + {}_3F_2(|\alpha|, |\beta|, P_1; \gamma, 2; 1) < 2 \qquad (0 \leqq k < \infty)$$

holds true, then

$$\mathcal{F}(\alpha, \beta, \gamma) f \in k\text{-}\mathcal{ST}.$$

Corollary 6. *Let γ be a real number such that*

$$\gamma > P_1 + 1 + |\alpha| + |\beta| \qquad (\alpha, \beta \in \mathbb{C} \setminus \{0\}),$$

where $P_1 = P_1(k)$ is given, as before, by (32). If, for some k $(0 \leqq k < \infty)$, $f \in k\text{-}\mathcal{ST}$ and the following hypergeometric inequality:

$$_4F_3\left(|\alpha| + 1, |\beta| + 1, P_1 + 1, 3; \gamma + 1, 2, 2; 1\right)$$

(41)
$$\leqq \frac{\gamma}{2(k+2)|\alpha\beta| P_1} \qquad (0 \leqq k < \infty)$$

holds true, then

$$\mathcal{F}(\alpha, \beta, \gamma) f \in k\text{-}\mathcal{UCV}.$$

3. Mapping and Inclusion Properties Involving the Function Classes $\mathcal{S}_\lambda^*$ and $\mathcal{C}_\lambda$

Just as in the work of Silverman [24, p. 110] on the familiar classes of starlike and convex functions of order μ $(0 \leqq \mu < 1)$, it is fairly straightforward to derive Lemma 5 and Lemma 6 involving the function classes $\mathcal{S}_\lambda^*$ and $\mathcal{C}_\lambda$ defined by (7) and (8), respectively.

Lemma 5. *Let $f \in \mathcal{A}$ be of the form* (1). *If*

(42)
$$\sum_{n=2}^{\infty} (\lambda + n - 1) |a_n| \leqq \lambda \qquad (\lambda > 0),$$

then $f \in \mathcal{S}_\lambda^*$.

Lemma 6. *Let $f \in \mathcal{A}$ be of the form* (1). *If*

$$(43) \qquad \sum_{n=2}^{\infty} n\left(\lambda + n - 1\right)\left|a_n\right| \leqq \lambda \qquad \left(\lambda > 0\right),$$

then $f \in \mathcal{C}_{\lambda}$.

Making use of Lemma 5 and Lemma 6, in conjunction with the coefficient inequalities (31) and (33), we now prove several mapping and inclusion properties for the convolution operator defined by (13), which involve the function classes $\mathcal{S}_{\lambda}^{*}$ and $\mathcal{C}_{\lambda}$.

Theorem 7. *Suppose that*

$$\alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \ldots, p), \qquad \Re\left(\beta_j\right) > 0 \quad (j = 1, \ldots, q),$$

and (in the case when $p = q + 1$)

$$\Re\left(\sum_{j=1}^{q} \beta_j\right) > P_1 - 1 + \sum_{j=1}^{p} \left|\alpha_j\right|,$$

where $P_1 = P_1\left(k\right)$ is given, as before, by (32). *If, for some k $\left(0 \leqq k < \infty\right)$, $f \in k\text{-}\mathcal{UCV}$ and the following hypergeometric inequality:*

(44)

$${}_{p+2}F_{q+2}\left(\left|\alpha_1\right|, \ldots, \left|\alpha_p\right|, P_1, \lambda + 1; \Re\left(\beta_1\right), \ldots, \Re\left(\beta_q\right), \lambda, 2; 1\right) < 2 \qquad \left(\lambda > 0\right)$$

holds true, then

$$I_{\beta_1, \ldots, \beta_q}^{\alpha_1, \ldots, \alpha_p} f \in \mathcal{S}_{\lambda}^{*}.$$

Proof. In view of Lemma 5, it suffices to show, for $f \in k\text{-}\mathcal{UCV}$ of the form (1), that

$$\sum_{n=2}^{\infty} \left(\lambda + n - 1\right)\left|a_n\right| \cdot \left|\Gamma_n\right| \leqq \lambda \qquad \left(\lambda > 0\right),$$

where Γ_n is defined by (15). Indeed, by applying the coefficient inequalities (31), we observe that

$$\sum_{n=2}^{\infty} (\lambda + n - 1) \, |a_n| \cdot |\Gamma_n|$$

$$\leqq \sum_{n=2}^{\infty} (\lambda + n - 1) \frac{(P_1)_{n-1}}{n!} \cdot \frac{(|\alpha_1|)_{n-1} \cdots (|\alpha_p|)_{n-1}}{(\Re(\beta_1))_{n-1} \cdots (\Re(\beta_q))_{n-1}} \cdot \frac{1}{(n-1)!}$$

$$= \sum_{n=1}^{\infty} (\lambda + n) \frac{(P_1)_n}{(n+1)!} \cdot \frac{(|\alpha_1|)_n \cdots (|\alpha_p|)_n}{(\Re(\beta_1))_n \cdots (\Re(\beta_q))_n} \cdot \frac{1}{n!}$$

$$= \lambda \left\{ {}_{p+2}F_{q+2} \left(|\alpha_1|, \ldots, |\alpha_p|, P_1, \lambda + 1; \Re(\beta_1), \ldots, \Re(\beta_q), \lambda, 2; 1 \right) - 1 \right\}$$

$$< \lambda \qquad (\lambda > 0),$$

by virtue of the hypothesis (44). This evidently completes the proof of Theorem 7.

Similarly, we can prove Theorem 8 below.

Theorem 8. *Suppose that*

$$\alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \ldots, p), \qquad \Re(\beta_j) > 0 \quad (j = 1, \ldots, q),$$

and (in the case when $p = q + 1$)

$$\Re \left(\sum_{j=1}^{q} \beta_j \right) > P_1 + \sum_{j=1}^{p} |\alpha_j|,$$

where $P_1 = P_1(k)$ is given, as before, by (32). If, for some k $(0 \leqq k < \infty)$, $f \in k\text{-}\mathcal{ST}$ and the following hypergeometric inequality:

(45)

$${}_{p+2}F_{q+2} \left(|\alpha_1|, \ldots, |\alpha_p|, P_1, \lambda + 1; \Re(\beta_1), \ldots, \Re(\beta_q), \lambda, 1; 1 \right) < 2 \qquad (\lambda > 0)$$

holds true, then

$$I_{\beta_1, \ldots, \beta_q}^{\alpha_1, \ldots, \alpha_p} f \in \mathcal{S}_\lambda^*.$$

In an analogous manner, Lemma 6 and the coefficient inequalities (31) and (33) would lead us to Theorem 9 and Theorem 10, respectively.

Theorem 9. *Suppose that*

$$\alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \ldots, p), \qquad \mathfrak{R}(\beta_j) > 0 \quad (j = 1, \ldots, q),$$

and (*in the case when* $p = q + 1$)

$$\mathfrak{R}\left(\sum_{j=1}^{q} \beta_j\right) > P_1 + \sum_{j=1}^{p} |\alpha_j|,$$

where $P_1 = P_1(k)$ *is given, as before, by* (32). *If, for some* k $(0 \leqq k < \infty)$, $f \in k\text{-}\mathcal{UCV}$ *and the following hypergeometric inequality* (45) *holds true, then*

$$I_{\beta_1,\ldots,\beta_q}^{\alpha_1,\ldots,\alpha_p} f \in \mathcal{C}_\lambda.$$

Theorem 10. *Suppose that*

$$\alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \ldots, p), \qquad \mathfrak{R}(\beta_j) > 0 \quad (j = 1, \ldots, q),$$

and (*in the case when* $p = q + 1$)

$$\mathfrak{R}\left(\sum_{j=1}^{q} \beta_j\right) > P_1 + 1 + \sum_{j=1}^{p} |\alpha_j|,$$

where $P_1 = P_1(k)$ *is given, as before, by* (32). *If, for some* k $(0 \leqq k < \infty)$, $f \in k\text{-}\mathcal{ST}$ *and the following hypergeometric inequality*:

$$(46) \quad {}_{p+3}F_{q+3}\left(|\alpha_1|, \ldots, |\alpha_p|, P_1, \lambda + 1, 2; \mathfrak{R}(\beta_1), \ldots, \mathfrak{R}(\beta_q), \lambda, 1, 1; 1\right) < 2$$

$$(\lambda > 0)$$

holds true, then

$$I_{\beta_1,\ldots,\beta_q}^{\alpha_1,\ldots,\alpha_p} f \in \mathcal{C}_\lambda.$$

For $f \in \mathcal{S}_\lambda^*$ of the form (1), Lemma 5 immediately yields the following coefficient inequalities:

$$(47) \qquad |a_n| \leqq \frac{\lambda}{\lambda + n - 1} \qquad (n \in \mathbb{N} \setminus \{1\}\,;\ \lambda > 0)\,.$$

Similarly, for $f \in \mathcal{C}_\lambda$ of the form (1), we have the coefficient inequalities:

$$(48) \qquad |a_n| \leqq \frac{\lambda}{n\,(\lambda + n - 1)} \qquad (n \in \mathbb{N} \setminus \{1\}\,;\ \lambda > 0)\,.$$

By applying the coefficient inequalities (47) and (48), in conjunction with Lemma 3 and Lemma 4, we can deduce *further* mapping and inclusion properties for the convolution operator defined by (13), which are associated with the function classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$. The details involved in the derivation of these mapping and inclusion properties are being left as an exercise for the interested reader.

Finally, we remark that each of the various results in this section (Theorems 7, 8, 9, and 10) can easily be restated, for $p = 2$ and $q = 1$, in terms of the Hohlov operator $\mathcal{F}\,(\alpha, \beta, \gamma)$ defined by (16). Furthermore, as we have already observed earlier, the interested reader should refer also to the closely-related *further* developments reported in the recent works by (for example) Ramachandran *et al.* ([20] and [21]), Srivastava *et al.* ([25] and [28]), and others. Remarkably, the Dziok-Srivastava convolution operator as well as the analytic function classes k-$\mathcal{ST}$ and k-$\mathcal{UCV}$ (together with many other interesting variants of these function classes k-$\mathcal{ST}$ and k-$\mathcal{UCV}$) are becoming increasingly popular in the recent as well as current literature in *Geometric Function Theory*.

Acknowledgements

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H. M. Srivastava

Department of Mathematics and Statistics

University of Victoria

Victoria, British Columbia V8W 3P4, Canada

E-Mail: harimsri@math.uvic.ca