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On a diophantine equation[1]

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Abstract

In this note we study the diophantine equation (1).

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In this note we study in positive integer numbers following the diophantine equation:

$$2^x + 5^y = z^2. \tag{1}$$

Theorem 1. *The diophantine equation (1) has exactly two solutions in nonnegative integers* $(x, y, z) \in \{(3, 0, 3), (2, 1, 3)\}$.

Proof. If $x = 0$, then we have the diophantine equation

$$5^y = y^2 - 1$$

or

$$(z - 1)(z + 1) = 5^y,$$

where $z - 1 = 5^u$ and $z + 1 = 5^{y-u}, y > 2u, u \in \mathbb{N}$.

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From here,we obtain:

$$5^{y-u} - 5^y = 2$$

or

$$5^u(5^{y-2u} - 1) = 2,$$

where $u = 0$ and $5^y = 3$, which is impossible.

If $y = 0$, then we have the diophantine equation

$$z^2 - 1 = 2^x$$

or

$$(z-1)(z+1) = 2^x,$$

where $z - 1 = 2^v$ and $z + 1 = 2^{x-v}, x > 2v, v \in \mathbb{N}$.

Form here, we obtain

$$2^{x-v} - 2^v = 2$$

or

$$2^v(2^{x-2v} - 1) = 2,$$

where $v = 1$ and $2^{x-2} = 2$, that is $v = 1$ and $x = 3$.
Therefore $x = 3, y = 0, z = 3$.

Now, we consider $x \geq 1$ and $y \geq 1$.

It follows from (1) that the number z is odd and it is not divisible by 5.

If $z \equiv \pm 1 (\text{mod } 5)$ then we have $z^2 \equiv 1 (\text{mod } 5)$ and if $z \equiv \pm 2 (\text{mod } 5)$ it results $z^2 \equiv 4 (\text{mod } 5) \equiv -1 (\text{mod } 5)$.

But, we have

$$2^{2k} = 4^k = (5-1)^k \equiv (-1)^k (\text{mod } 5)$$

and

$$2^{2k+1} = 2 \cdot 4^k \equiv 2 \cdot (-1)^k (\text{mod } 5), k \in \mathbb{N}.$$

It results that the number x is even.

Now, we consider $x = 2k, k \in \mathbb{N}$ .From (1) we have

$$z^2 - 2^{2k} = 5^y$$

or

$$(z - 2^k)(z + 2^k) = 5^y,$$

where $z - 2^k = 5^w$ and $z + 2^k = 5^{y-w}, y > 2w$. From here, we obtain

$$5^w(5^{y-2w} - 1) = 2^{k+1}$$

which implies $w = 0$ and

$$5^y - 2^{k+1} = 1. \tag{2}$$

The diophantine equation (2) is a diophantine equation by Catalan's type

$$a^b - c^d = 1$$

which has in positive integer numbers (> 1) only the solutions $a = 3, b = 2, c = 2$ and $d = 3([3], [6], [7])$.

It results the diophantine equation (2) has the solution only if $y = 1$. Then we have $2^{k+1} = 2^2$, where $k = 1$. Therefore $x = 2, y = 1, z = 3$.

In concluding, the diophantine equation (1) has the solutions:$(x, y, z) \in \{(3, 0, 3), (2, 1, 3)\}$.

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