General Mathematics Vol. 15, No. 4 (2007), 7-15

Certain sufficient conditions for univalence [1]

Narayanasamy Seenivasagan and Daniel Breaz

Abstract

In this paper, we determined conditions on β, α_i and $f_i(z)$ so that the integral operator $\left\{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} (\frac{f_i(t)}{t})^{\frac{1}{\alpha_i}} dt\right\}^{\frac{1}{\beta}}$ is univalent in the open unit disk for the two subclasses analytic functions.

2000 Mathematical Subject Classification: Primary 30C80, Secondary 30C45.

Key words: Integral operator, univalent functions, Schwartz's Lemma.

1 Introduction

Let $\mathcal{A}$ be the class of all analytic functions $f(z)$ defined in the open unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in U. Let $\mathcal{A}_2$ be the subclass of $\mathcal{A}$ consisting of functions is of the form

$$f(z) = z + \sum_{k=3}^{\infty} a_k z^k. \tag{1.1}$$

[1] *Received 12 September, 2007*

Accepted for publication (in revised form) 15 October, 2007

Let T be the univalent [6] subclass of $\mathcal{A}$ which satisfies

$$\left|\frac{z^2f'(z)}{(f(z))^2}-1\right|<1 \quad (z\in U). \tag{1.2}$$

Let T_2 be the subclass of T for which $f''(0)=0$. Let $T_{2,\mu}$ be the subclass of T_2 consisting of functions is of the form (1.1) which satisfy

$$\left|\frac{z^2f'(z)}{(f(z))^2}-1\right|\le\mu \quad (z\in U) \tag{1.3}$$

for some μ $(0<\mu\le 1)$, and let us denote $T_{2,1}\equiv T_2$. Furthermore, for some real p with $0<p\le 2$ we define a subclass $\mathcal{S}(p)$ of $\mathcal{A}$ consisting of all function $f(z)$ which satisfy

$$\left|\left(\frac{z}{f(z)}\right)''\right|\le p \quad (z\in U).$$

Singh [5] has shown that if $f(z)\in\mathcal{S}(p)$, then $f(z)$ satisfies

$$\left|\frac{z^2f'(z)}{(f(z))^2}-1\right|\le p|z|^2, \quad (z\in U). \tag{1.4}$$

Pascu [2] has proved the following theorem:

Theorem 1.1. *[2, 3] Let $\beta\in\mathbb{C}$, $Re\beta\ge\gamma>0$. If $f\in\mathcal{A}$ satisfies*

$$\frac{1-|z|^{2\gamma}}{\gamma}\left|\frac{zf''(z)}{f'(z)}\right|\le 1, \quad (z\in U),$$

then the integral operator

$$F_\beta(z)=\left[\beta\int_0^z t^{\beta-1}f'(t)dt\right]^{\frac{1}{\beta}}$$

is in $f\in\mathcal{S}$.

Theorem 1.2. *[4] Let $\alpha,\beta\in\mathbb{C}$ and $Re\beta\ge Re\alpha\ge\frac{3}{|\alpha|}$. Let $f\in\mathcal{A}$, that satisfies the condition*

$$\left|\frac{z^2f'(z)}{(f(z))^2}-1\right|<1, \quad (z\in U)$$

and $|f(z)| \leq 1, \quad (z \in U)$, *then the integral operator*

$$H_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t} \right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}}$$

is in $\mathcal{S}$.

Using Theorem 1.1 and Theorem 1.2, Breaz and Breaz [1] obtained the following Theorems.

Theorem 1.3. *[1] Let* $\alpha, \beta \in \mathbb{C}$ *and* $Re\beta \geq Re\alpha > \frac{3n}{|\alpha|}$. *Let* $f_i \in T_2$ *and*

$$f_i(z) = z + \sum_{k=3}^{\infty} a_k^i z^k \tag{1.5}$$

for all $i = 1, 2, \cdots, n$, $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ *and if*

$$|f_i(z)| \leq 1, \quad (z \in U, \quad i = 1, 2, \cdots, n),$$

then the integral operator

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}} \tag{1.6}$$

is in $\mathcal{S}$.

Theorem 1.4. *[1] Let* $\alpha, \beta \in \mathbb{C}$ *and* $Re\beta \geq Re\alpha > \frac{n(\mu+2)}{|\alpha|}$. *Let* $f_i \in T_{2,\mu}$ *defined by (1.5) for all* $i = 1, 2, \cdots, n$, $n \in \mathbb{N}^*$ *and if* $|f_i(z)| \leq 1, \quad (z \in U, \quad i = 1, 2, \cdots, n)$, *then the integral operator defined by (1.6) is in* $\mathcal{S}$.

Theorem 1.5. *[1] Let* $\alpha, \beta \in \mathbb{C}$ *and* $Re\beta \geq Re\alpha > \frac{n(p+2)}{|\alpha|}$. *Let* $f_i \in \mathcal{S}(p)$ *defined by (1.5) for all* $i = 1, 2, \cdots, n$, $n \in \mathbb{N}^*$ *and if* $|f_i(z)| \leq 1, \quad (z \in U, \quad i = 1, 2, \cdots, n)$, *then the integral operator defined by (1.6) is in* $\mathcal{S}$.

Theorem 1.2 is true even if $Re\beta \geq Re\alpha \geq 3/|\alpha|$ is replaced by the condition $Re\beta \geq 3/|\alpha|$. Similarly Theorem 1.3 is true even if $Re\beta \geq Re\alpha \geq$

$3n/|\alpha|$ is replaced by the condition $Re\beta \geq 3n/|\alpha|$, Theorem 1.4 is true even if $Re\beta \geq Re\alpha \geq \frac{n(\mu+2)}{|\alpha|}$ is replaced by the condition $Re\beta \geq \frac{n(\mu+2)}{|\alpha|}$ and Theorem 1.5 is true even if $Re\beta \geq Re\alpha \geq \frac{n(p+2)}{|\alpha|}$ is replaced by the condition $Re\beta \geq \frac{n(p+2)}{|\alpha|}$.

In this paper we extend Theorems 1.3-1.5 and also obtain the sufficient condition for univalency of certain integral operator.

To prove our main results we need the following lemma:

Lemma 1.1. *(Schwarz's Lemma) If the function $w(z)$ is analytic in the unit desk U, $w(0) = 0$, and $|w(z)| \leq 1$, for all $z \in U$, then*

$$|w(z)| \leq |z|, \quad (z \in U)$$

and equality holds only if $w(z) = \epsilon z$, where $|\epsilon| = 1$.

2 Sufficient Conditions For Univalence

For $f_i \in \mathcal{A}_2$ $(i = 1, 2, \cdots, n)$ and $\alpha_1, \alpha_2, \cdots, \alpha_n, \beta \in \mathbb{C}$, we define an integral operator by

$$F_{\alpha_1,\alpha_2,\cdots,\alpha_n,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\frac{1}{\alpha_i}} dt \right\}^{\frac{1}{\beta}}. \tag{2.1}$$

When $\alpha_i = \alpha$ for all $i = 1, 2, \cdots, n$, $F_{\alpha_1,\alpha_2,\cdots,\alpha_n,\beta}(z)$ becomes the integral operator $F_{\alpha,\beta}(z)$ considered in Theorem 1.3.

Theorem 2.1. *Let $M \geq 1$, $f_i \in T_{2,\mu_i}$ defined by (1.5), $\alpha_i, \beta \in \mathbb{C}$, $Re\beta \geq \gamma$ and*

$$\gamma := \sum_{i=1}^{n} \frac{(1+\mu_i)\, M + 1}{|\alpha_i|} \quad (0 < \mu_i \leq 1,\ for\ all\ i = 1, 2, \cdots, n,\ n \in \mathbb{N}^*). \tag{2.2}$$

If

$$|f_i(z)| \leq M, \quad (z \in U, \quad i = 1, 2, \cdots, n),$$

then the integral operator $F_{\alpha_1,\alpha_2,\cdots,\alpha_n,\beta}(z)$ defined by (2.1) is in $\mathcal{S}$.

Proof. Define a function

$$h(z)=\int_0^z \prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}} dt,$$

then we have $h(0)=h'(0)-1=0$. Also a simple computation yields

$$h'(z)=\prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\frac{1}{\alpha_i}}$$

and

$$\frac{zh''(z)}{h'(z)}=\sum_{i=1}^{n}\frac{1}{\alpha_i}\left(\frac{zf_i'(z)}{f_i(z)}-1\right). \tag{2.3}$$

From equation (2.3), we have

$$\begin{aligned}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)\\ &= \sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left(\left|\frac{z^2f_i'(z)}{(f_i(z))^2}\right|\left|\frac{f_i(z)}{z}\right|+1\right)\end{aligned} \tag{2.4}$$

From the hypothesis, we have $|f_i(z)|\leq M \quad (z\in U, \quad i=1,2,\cdots,n)$, then by Schwarz Lemma, we obtain that

$$|f_i(z)|\leq M|z| \quad (z\in U, \quad i=1,2,\cdots,n).$$

We apply this result in inequality (2.4), we obtain

$$\begin{aligned}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left(\left|\frac{z^2f_i'(z)}{(f_i(z))^2}\right|M+1\right)\\ &\leq \sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left(\left|\frac{z^2f_i'(z)}{(f_i(z))^2}-1\right|M+M+1\right)\\ &= \sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left(\mu_iM+M+1\right)=\sum_{i=1}^{n}\frac{\left(1+\mu_i\right)M+1}{|\alpha_i|}.\end{aligned} \tag{2.5}$$

Because of $f_i \in T_{2,\mu_i}$, (1.3) in (2.5) and in view of (2.2) we have

$$\left|\frac{zh''(z)}{h'(z)}\right| < \sum_{i=1}^{n} \frac{(1+\mu_i)\, M + 1}{|\alpha_i|} = \gamma. \tag{2.6}$$

Multiply (2.6) by

$$\frac{1-|z|^{2\gamma}}{\gamma},$$

we have

$$\frac{1-|z|^{2\gamma}}{\gamma}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1-|z|^{2\gamma} < 1 \quad (z \in U).$$

Since $Re\beta \geq \gamma > 0$ it follows from Theorem 1.1 that

$$\left[\beta \int_0^z t^{\beta-1} h'(t)dt\right]^{\frac{1}{\beta}} \in \mathcal{S}.$$

Since

$$\left[\beta \int_0^z t^{\beta-1} h'(t)dt\right]^{\frac{1}{\beta}} = \left[\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}} dt\right]^{\frac{1}{\beta}} = F_{\alpha_1,\alpha_2,\cdots,\alpha_n,\beta}(z),$$

the integral operator $F_{\alpha_1,\alpha_2,\cdots,\alpha_n,\beta}(z)$ defined by (2.1) is in $\mathcal{S}$.

Remark 2.1. *By taking $M = 1$, $\alpha_i = \alpha$, $for\ all$ $i = 1, 2, \cdots, n$, then Theorem 2.1 reduces to Theorem 1.4. By taking $\mu_i = \mu = 1$, $\alpha_i = \alpha$, $for\ all$ $i = 1, 2, \cdots, n$, then Theorem 2.1 reduces to Theorem 1.3.*

Theorem 2.2. *Let $M \geq 1$, $f_i \in \mathcal{S}(p)$ defined by (1.5), $\alpha_i, \beta \in \mathbb{C}$, $Re\beta \geq \gamma_1$ and*

$$\gamma_1 := \sum_{i=1}^{n} \frac{(1+p)\, M + 1}{|\alpha_i|} \quad (\ for\ all\ i = 1, 2, \cdots, n,\ n \in \mathbb{N}^*). \tag{2.7}$$

If

$$|f_i(z)| \leq M \quad (z \in U, \quad i = 1, 2, \cdots, n),$$

then the integral operator $F_{\alpha_1,\alpha_2,\cdots,\alpha_n,\beta}(z)$ defined by (2.1) is in $\mathcal{S}$.

Proof. Define a function

$$h(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i'(t)}{t}\right)^{\frac{1}{\alpha_i}} dt,$$

then we have $h(0) = h'(0) - 1 = 0$. Because of $f_i \in \mathcal{S}(p)$, (1.4) in (2.5), in view of (2.7) we have

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^n \frac{1 + M + Mp|z|^2}{|\alpha_i|} \tag{2.8}$$

$$< \sum_{i=1}^n \frac{(1+p)\,M + 1}{|\alpha_i|} = \gamma_1 \quad (z \in U). \tag{2.9}$$

Rest of the proof is similar to Theorem 2.1, then we omit the details.

Remark 2.2. *By taking $M = 1$, $\alpha_i = \alpha$, for all $i = 1, 2, \cdots, n$, then Theorem 2.2 reduces to Theorem 1.5.*

Theorem 2.3. *Let $\alpha_i, \beta \in \mathbb{C}$, $Re\beta \geq \gamma_2$ and*

$$\gamma_2 := \sum_{i=1}^n \frac{\beta_i}{|\alpha_i|} \quad (0 < \beta_i \leq 1,\ for\ all\ i = 1, 2, \cdots, n,\ n \in \mathbb{N}^*). \tag{2.10}$$

If $f_i \in \mathcal{A}_2$ defined by (1.5) satisfy the conditions

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq \beta_i \quad (0 < \beta_i \leq 1,\ z \in U, \quad i = 1, 2, \cdots, n), \tag{2.11}$$

then the integral operator $F_{\alpha_1, \alpha_2, \cdots, \alpha_n, \beta}(z)$ defined by (2.1) is in $\mathcal{S}$.

Proof. From (2.3), we get

$$\left|\frac{zh''(z)}{h'(z)}\right| = \left|\sum_{i=1}^n \frac{1}{\alpha_i}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right| \leq \sum_{i=1}^n \frac{1}{|\alpha_i|}\left|\frac{zf_i'(z)}{f_i(z)} - 1\right|. \tag{2.12}$$

Substituting (2.11) in (2.12) and in view of (2.10) we have

$$\left|\frac{zh''(z)}{h'(z)}\right| < \sum_{i=1}^n \frac{\beta_i}{|\alpha_i|} = \gamma_2. \tag{2.13}$$

Rest of the proof is similar to Theorem 2.1, then we omit the details.

By taking $\beta_i = 1$ and $\alpha_i = \alpha$ $(for\ all\ i = 1, 2, \cdots, n)$ in Theorem 2.3, we obtained the following result.

Example 2.1. *Let $\alpha, \beta \in \mathbb{C}$, $Re\beta \geq \frac{n}{|\alpha|}$. If $f_i \in \mathcal{A}_2$ defined by (1.5) satisfy the conditions*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq 1 \quad (z \in U, \quad i = 1, 2, \cdots, n), \tag{2.14}$$

then the integral operator $F_{\alpha,\beta}(z)$ defined by (1.6) is in $\mathcal{S}$.

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Narayanasamy Seenivasagan
2A Kamarajar Street
Samatharmapuram, Theni
625531 India
E-mail: vasagan2000@yahoo.co.in

Daniel Breaz
Department of Mathematics and Computer Science
"1 Decembrie 1918" University of Alba Iulia
510009, str. N. Iorga, No. 11-13, Alba, Romania
E-mail: dbreaz@uab.ro