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On the law of large numbers for free identically distributed random variables[1]

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Abstract

A version of law of large numbers for free identically distributed random variables is considering at this work. It shown that

$$\lim_{t\to\infty} t\,\mu\,(x:\ |x|>t)=0$$

is a sufficient and necessary condition for the weak law of large numbers for the sequence $X_1, X_2, ...,$ free random variables.

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1 Introduction

Analytic theory of free additive convolution is useful in frame of this article. The calculation of free additive convolution is based on an analoque of the Fourier transform first introduced by Voiculescu [6]. We need the version of

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this apparatus which is suitable for the convolution of arbitrary probability measures [1].

First, some notation. Let $\mathbb{C}$ denote the complex field, $\mathbb{C}^+$ and $\mathbb{C}^-$ the upper and lower half plane. We consider

$$\Gamma_\alpha = \{z = x + iy \ : \ y > 0 \text{ and } |x| < \alpha y\} \ ,$$

$$\Gamma_{\alpha,\beta} = \{z \in \Gamma_\alpha \ : \ y > \beta\} \ , \ \alpha, \beta > 0 \ .$$

where α and β are positive numbers.

Given a probability measure μ on $\mathbb{R}$, its Cauchy transform $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ is defined as

$$G_\mu(z) := \int\limits_{-\infty}^{\infty} \frac{\mu \, \mathrm{d}x}{z - x} = \mathbb{E}\left((z - X)^{-1}\right)$$

The reciprocal Cauchy transform is defined by $F_\mu : \mathbb{C}^+ \to \mathbb{C}^+$, $F_\mu(z) = 1/G_\mu(z)$.

2 Proof of the main result

First, we would like to formulate the theorem in terms of free convolutions rather than random variables. To do this, observe that given a self-adjoint random variable X affiliated with some algebras $\mathcal{A}$ and a scalar $\lambda > 0$, we have

$$\mu_{\lambda X} = D_\lambda \mu_X$$

where D_λ is the dilation of a measure μ defined by $D_\lambda \mu(A) = \mu(\lambda^{-1} A)$, ($A \subset \mathbb{R}$ measurable).

Theorem 2.1. *Let μ be a probability measure $\mathbb{R}$. The following conditions are equivalent:*

(i) *There exist real constants $M_1, M_2, ... \in \mathbb{R}$ such that*

$$\lim_{n\to\infty} \nu_n = \delta_0 \ ,$$

where $\nu_n = \underbrace{D_{1/n}\mu \otimes ... \otimes D_{1/n}\mu}_{n-\text{ori}} \otimes \delta_{-M_n}$ *and* δ_0 *Dirac distribution;*

(ii) *The measure μ satisfies $\lim\limits_{t\to\infty} t\, \mu\,(x : \, |x| > t) = 0$. Moreover, if (ii) is satisfied, the constants M_n in (i) can be chosen to be $M_n = \int\limits_{-n}^{+n} t \mathrm{d}\mu(t)$.*

For the proof of this theorem, we establish some preparatory lemmas. The foolowing result is related by Bercovici and Voiculescu (1993) in Proposition 4.5. to (1993,[7]).

Lemma 2.1. ([7]) *Let μ be a probability measure on $\mathbb{R}$. Given a truncated cone $\Gamma_{\alpha,\beta}$. Then exists a truncated cone $\Gamma_{\alpha',\beta'}$ such that $F_\mu(\Gamma_{\alpha',\beta'}) \subset \Gamma_{\alpha,\beta}$.*

Proof. Fix a number $\alpha' \in (0,1)$ such that $\gamma = \frac{1}{\tan\alpha'}$, and choose $\beta' > 0$ so large that

$$|F_\mu(u) - u| \leq \sin\gamma \cdot |u| \ , \tag{2.1}$$

for all $\Im u > \beta'$ and $\beta' > \frac{\beta}{1-\alpha'}$. The relation (2.1) is possible beacause $uG_\mu(u) = \int\limits_{\mathbb{R}} \frac{z}{z-t}\, \mathrm{d}\mu(t) \overset{|u|\to\infty}{\longrightarrow} 1$, thus $\frac{F_\mu(u)}{u} \overset{|u|\to\infty}{\longrightarrow} 1$, forall $u \in \Gamma_{\alpha'}$.

We note the disk $D_u = \{w \ \ |w - u| < \sin\gamma \cdot |u|\}$. We observe that $F_\mu(u) \in D_u$ if $u \in \Gamma_{\alpha',\beta'} \subset \Gamma_{\alpha'}$, while the implication $D_u \subset \Gamma_{\alpha,\beta}$ if $u \in \Gamma_{\alpha',\beta'}$ is justify in figure 1, where $u' \in \Gamma_{\alpha,\beta}$, while $\Im u > \Im u'$

In the sequel we will use the following notation. If $y \geq 0$, we denote $I_y = [-y, y]$ and $\Delta_y = (-\infty, -y) \cup (y, +\infty)$.

Proposition 2.1. ([8]) *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of probability measures on $\mathbb{R}$. The following assertions are equivalent*

(a) The sequence $\{\mu_n\}_{n=1}^{\infty}$ converges weakly to a probability measure μ;

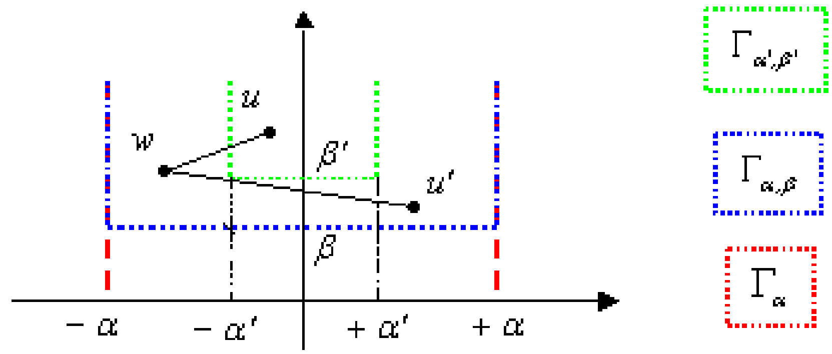

Figure 1: Graphic justfy of implication $D_u \subset \Gamma_{\alpha,\beta}$, $u \in \Gamma_{\alpha'}$

(b) *There exist* $\alpha, \beta > 0$ *such that the sequence* $(\psi_{\nu_n})_{n=\overline{1,\infty}} \longrightarrow \psi$, $\psi \in \Gamma_{\alpha,\beta}$ *and* $\psi_{\nu_n}(u) = 0(|u|)$ *if* $u \to \infty$, $u \in \Gamma_{\alpha,\beta}$*;*

(c) *There exist* $\alpha', \beta' > 0$ *such that the functions* ψ_{μ_n} *are defined on* $\Gamma_{\alpha',\beta'}$ *for every* n, $\lim\limits_{n\to\infty} \psi_{\mu_n}(iy)$ *exists for every* $y > \beta'$ *and* $\psi_{\mu_n}(iy) = o(y)$ *uniformly in* n *as* $y \to \infty$.

Lemma 2.2. *Let* μ *be a probability measure on* $\mathbb{R}$ *satisfying condition (i) of Theorem 2.1. Then*

$$\lim_{y\to\infty} (\Im F_\mu(iy) - y) = 0 \, .$$

Proof. By Proposition 2.1,(ii) weak convergence of ν_0 to δ_0 implies that there exist α_0, $\beta_0 > 0$ such that $\psi_{\nu_n}(u) \to 0$ for $u \in \Gamma_{\alpha_0,\beta_0}$.

However, $\psi_{\nu_n}(u) = n\psi_{D_{1/n}\mu}(u) - M_n = n \cdot \frac{1}{n}\psi_\mu(nu) - M_n = \psi_\mu(nu) - M_n$, which implies

$$\lim_{n\to\infty} \Im\psi_{\nu_n}(u) = \lim_{n\to\infty} \Im\psi_\mu(nu) = 0 \, , \tag{2.2}$$

for all $u \in \Gamma_{\alpha_0,\beta_0}$. By Lemma 2.1, exists $\alpha_2, \alpha_3, \beta_2, \beta_3 > 0$ such that $\alpha_0 > \alpha_2$ and $\beta_0 < \beta_2$ such that F_μ has an inverse on $\Gamma_{\alpha_2,\beta_2}$ and $\Gamma_{\alpha_0,\beta_0} \supset \Gamma_{\alpha_2,\beta_2} \supset F_\mu(\Gamma_{\alpha_3,\beta_3})$,($\alpha_2 > \alpha_3$ and $\beta_3 > \beta_2$). Therefore for any *for all* $u \in \Gamma_{\alpha_3,\beta_3}$ it follows that

$$F_\mu(u) = u - \psi_\mu(F_\mu(u)) \, . \tag{2.3}$$

Indeed, in condition $\psi_\mu(u) = F^{-1}(u) - u$ replace $u \to F_\mu(u)$ and we obtain eager relation. In particular, defining α_1, β_1 such that $\alpha_0 > \alpha_1 > \alpha_2$ and $\beta_0 < \beta_1 < \beta_2$ then the relation (2.2) holds $\forall\, u \in \Gamma_{\alpha_1,\beta_1}$(because $\Gamma_{\alpha_0,\beta_0} \supset \Gamma_{\alpha_1,\beta_1}$). Since $\bigcup\limits_{n=1}^{\infty} n\Gamma_{\alpha_1,\beta_1} = \Gamma_{\alpha_0,\beta_0}$, it is immediate now that

$$\lim_{|u|\to\infty\,;\, u\in\Gamma_{\alpha_1,\beta_1}} \Im\psi_\mu(u) = 0 \, . \tag{2.4}$$

Since $\frac{F_\mu(u)}{u} \stackrel{|u|\to\infty}{\to} 1$, $u \in \Gamma_\alpha$, we conclude that $F_\mu(iy) \to iy$, and hence $|F_\mu(iy)| \stackrel{y\to\infty}{\to} \infty$. Then $\lim\limits_{y\to\infty} \Im\psi_\mu(F_\mu(iy)) = 0$ (in relation (2.4) replace u by $F_\mu(iy)$). If in (2.3) we replace u by iy, we obtain $\Im F_\mu(iy) - y = -\Im\psi_\mu(F_\mu(iy))$, which concludes the proof.

Lemma 2.3. *Let σ be a finite positive measure on $\mathbb{R}$ such that $\lim\limits_{y\to\infty} y\sigma(\Delta_y) = 0$. Then the following hold*

(i) $\lim\limits_{y\to\infty} y\sqrt{y} \int\limits_{-\infty}^{+\infty} \frac{|t|}{y^2+t^2}\, \mathrm{d}\sigma(t) = 0$;

(ii) $\lim\limits_{y\to\infty} \frac{1}{\ln y} \int\limits_{-y}^{+y} |t|\, \mathrm{d}\sigma(t) = 0$;

(iii) $\lim\limits_{y\to\infty} \frac{1}{y^k} \int\limits_{-y}^{+y} |t|^{k+1}\, \mathrm{d}\sigma(t) = 0\, , k > 0$.

Proof. Consider the function

$$f_y(t) = \begin{cases} \frac{y\sqrt{y}|t|}{y^2+t^2} \, , & t \in [-\sqrt{y}, \sqrt{y}] \\ 0\, , & \text{otherwise.} \end{cases}$$

Then $\lim\limits_{y\to\infty} f_y(t) = 0$, *for all* $t \in \mathbb{R}$ and for all $y > 0$ we have

$$|f_y(t)| = \left|\frac{y\sqrt{y}\,|t|}{y^2+t^2}\right| \le \frac{y^2}{y^2+y} \le 1 \in \mathcal{L}^1(\sigma)\ .$$

This condition is the condition of the Lebesque convergence theorem (2000,[4], Theoreme 3.8., page 76), which implies that $\lim\limits_{y\to\infty} \int\limits_{-\infty}^{+\infty} f_y(t)\,\mathrm{d}\sigma(t) = 0$.

The function $g(t) = \frac{|t|}{y^2+t^2}$ is increasing for $t \in [0, \sqrt{y}]$ and $y > 1$.

The result now follows because

$$\text{(2.5)}\qquad y\sqrt{y}\int\limits_{-\infty}^{+\infty} \frac{|t|}{y^2+t^2}\,\mathrm{d}\sigma(t) = \int\limits_{-\sqrt{y}}^{\sqrt{y}} f_y(t)\,\mathrm{d}\sigma(t) + y\sqrt{y}\int\limits_{\Delta_{\sqrt{y}}} \frac{|t|}{y^2+t^2}\,\mathrm{d}\sigma(t)\ .$$

But $y\sqrt{y}\int\limits_{\Delta_{\sqrt{y}}} \frac{|t|}{y^2+t^2}\,\mathrm{d}\sigma(t) \le \frac{1}{2}\sqrt{y}\sigma(\Delta_{\sqrt{y}})$ if to count on inequality $\frac{\sqrt{y}}{y+1} \le \frac{1}{2}$.

In condition (2.5) through on limit when $y \to \infty$,

$$\lim_{y\to\infty} y\sqrt{y}\int\limits_{-\infty}^{+\infty} \frac{|t|}{y^2+t^2}\,\mathrm{d}\sigma(t) = 0 + \frac{1}{2}\lim_{y\to\infty}\sqrt{y}\sigma(\Delta_{\sqrt{y}}) = 0\ .$$

To prove (ii) we integrate by parts and we count on $\sigma(\Delta_y) = \sigma(\mathbb{R}) - \sigma(I_y) = 1 - \sigma([-y,y]) = 1 - 2y$, we then have $\sigma(\Delta_t) = 1 - 2\underbrace{\sigma(t)}_{t}$.

We have

$$\begin{aligned}
\int\limits_{I_y} |t|\ \mathrm{d}\sigma(t) &= 2\int\limits_0^y |t|\ \mathrm{d}\sigma(t) = 2\left(t\sigma(t)|_0^y - \int\limits_0^y \sigma(t)\,\mathrm{d}t\right)\\
&= 2y\sigma(y) - 2\int\limits_0^y \sigma(t)\,\mathrm{d}t\\
&= [1-\sigma(\Delta_y)]\,\sigma(y) + \int\limits_0^y \sigma(\Delta_t)\,\mathrm{d}t - y\\
&= -y\sigma(\Delta_y) + \int\limits_0^y \sigma(\Delta_t)\,\mathrm{d}t\ .
\end{aligned}$$

Then

$$\frac{1}{\ln y}\int_{I_y} |t|\ \mathrm{d}\sigma(t) = \frac{-y}{\ln y}\sigma(\Delta_y) + \frac{1}{\ln y}\int_0^y \sigma(\Delta_t)\ \mathrm{d}t\ .$$

It is clear that $\frac{-y}{\ln y}\sigma(\Delta_y) = o(1)$ if $y \to \infty$. Select $\epsilon > 0$ and choose $N > 0$ large enough such that $t\sigma(\Delta_t) < \epsilon$, $\forall\, t \geq N$. Then, as $y \to \infty$,

$$\begin{aligned}\frac{1}{\ln y}\int_0^y \sigma(\Delta_t)\ \mathrm{d}t &= \frac{1}{\ln y}\int_0^N \sigma(\Delta_t)\ \mathrm{d}t + \frac{1}{\ln y}\int_N^y \frac{t\sigma(\Delta_t)}{t}\ \mathrm{d}t \\ &\leq \frac{N\sigma(\mathbb{R})}{\ln y} + \frac{1}{\ln y}\int_N^y \frac{\epsilon}{t}\ \mathrm{d}t \\ &\leq \epsilon + o(1)\ .\end{aligned}$$

We will use the notation $M_y = \int_{I_y} t\ \mathrm{d}\mu(t)$ if $y \in \mathbb{Z}$.

Lemma 2.4. *Let μ be a probability measure on $\mathbb{R}$ such that $\lim_{y\to\infty} y\mu(\Delta_y) = 0$. Then $F_\mu(iy) = iy - M_y + o(1)$ as $y \to \infty$.*

Proof. We will prove the estimate $G_\mu(iy) = \frac{1}{iy} - \frac{M_y}{y^2} + \frac{1}{y^2}\,o(1)$ as $y \to \infty$. We can estimate separately the real and the imaginary parts of $G_\mu(iy)$. For the real part we have

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\mu(t)}{z-t} \Longrightarrow G_\mu(iy) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\mu(t)}{iy-t} = \int_{-\infty}^{+\infty} \frac{iy+t}{-y^2-t^2}\ \mathrm{d}\mu(t)\ ,$$

thus

$$\begin{aligned}\Re G_\mu(iy) &= \int_{-\infty}^{+\infty} \frac{-t}{y^2+t^2}\ \mathrm{d}\mu(t) = \int_{I_y} \frac{-t}{y^2+t^2}\ \mathrm{d}\mu(t) + \int_{\Delta_y} \frac{-t}{y^2+t^2}\ \mathrm{d}\mu(t) \\ &= \int_{I_y} \frac{-ty^2-t^3}{y^2(y^2+t^2)}\ \mathrm{d}\mu(t) + \int_{I_y} \frac{t^3}{y^2(y^2+t^2)}\ \mathrm{d}\mu(t) + \int_{\Delta_y} \frac{-t}{y^2+t^2}\ \mathrm{d}\mu(t) \\ &= -\frac{M_y}{y^2} + \int_{I_y} \frac{t^3}{y^2(y^2+t^2)}\ \mathrm{d}\mu(t) + \int_{\Delta_y} \frac{-t}{y^2+t^2}\ \mathrm{d}\mu(t)\ .\end{aligned}$$

On the other side,

$$\left|\int\limits_{I_y} \frac{t^3}{y^2(y^2+t^2)}\,\mathrm{d}\mu(t) + \int\limits_{\Delta_y} \frac{-t}{y^2+t^2}\,\mathrm{d}\mu(t)\right| \le \int\limits_{I_y} \frac{|t|^3}{y^2(y^2+t^2)}\,\mathrm{d}\mu(t) +$$

$$\int\limits_{\Delta_y} \frac{|t|}{y^2+t^2}\,\mathrm{d}\mu(t) \le \underbrace{\frac{1}{y^4}\int\limits_{I_y} |t|^3\,\mathrm{d}\mu(t)}_{\frac{1}{y^2}\cdot\frac{1}{y^2}\int\limits_{I_y}|t|^{2+1}\,\mathrm{d}\mu(t)} + \frac{1}{2y}\underbrace{\int\limits_{\Delta_y}\mathrm{d}\mu(t)}_{\mu(\Delta_y)}$$

$$\overset{\text{Lemma 2.3 }(iii)}{\le} \frac{1}{y^2}\,o(1) + \frac{1}{2y^2}\,y\mu(\Delta_y) = \frac{1}{y^2}\,o(1)\,.$$

Thus, $\Re G_\mu(iy) \le -\frac{M_y}{y^2} + \frac{1}{y^2}\,o(1)$. The imaginary part

$$\begin{aligned}
\Im G_\mu(iy) &= \int\limits_{-\infty}^{+\infty} \frac{-y}{y^2+t^2}\,\mathrm{d}\mu(t) = -\frac{1}{y}\int\limits_{-\infty}^{+\infty} \frac{y^2 \pm t^2}{y^2+t^2}\,\mathrm{d}\mu(t)\\
&= -\frac{1}{y}\underbrace{\int\limits_{-\infty}^{+\infty}\mathrm{d}\mu(t)}_{\mu(\mathbb{R})=1} + \frac{1}{y}\int\limits_{-\infty}^{+\infty}\frac{t^2}{y^2+t^2}\,\mathrm{d}\mu(t)\\
&= -\frac{1}{y} + \frac{1}{y}\int\limits_{I_y}\frac{t^2}{y^2+t^2}\,\mathrm{d}\mu(t) + \frac{1}{y}\int\limits_{\Delta_y}\frac{t^2}{y^2+t^2}\,\mathrm{d}\mu(t)\\
&\le -\frac{1}{y} + \frac{1}{y^3}\int\limits_{I_y} t^2\,\mathrm{d}\mu(t) + \frac{1}{2y}\,\mu(\Delta_y)\\
&\overset{\text{Lemma 2.3 }(iii)}{\le} -\frac{1}{y} + \frac{1}{y^2}\,o(1)\,.
\end{aligned}$$

But $\frac{1}{G_\mu(iy)} = -\frac{y^2}{iy+M_y-o(1)} = iy\cdot\frac{iy}{iy+M_y-o(1)} = iy\left(1+\frac{M_y-o(1)}{iy}\right)^{-1}$.

We say the development in Mac-Laurent series of function $x \mapsto \frac{1}{1+x} \approx 1 - x + ...$ Then the reciprocal Cauchy transform is $F_\mu(iy) = \frac{1}{G_\mu(iy)} = iy\left(1-\frac{M_y-o(1)}{iy}\right) = iy - M_y + o(1)$.

Lemma 2.5. *Let μ be a probability measure on $\mathbb{R}$ such that $\lim\limits_{y\to\infty} y\mu(\Delta_y) = 0$ and $z \in \Gamma_{1/4}$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}z} F_\mu(z) = 1 + \frac{1}{\sqrt{|z|}}\, o(1) \ ,$$

as $|z| \to \infty$ in $\Gamma_{1/4}$.

Proof. By the Nevanlinna representation according to (1963,[2]) of $F_\mu(z)$,

$$F(z) = a + z + \int\limits_{-\infty}^{\infty} \frac{1+tz}{t-z}\, \mathrm{d}\sigma(t) \ , \ z \in \mathbb{C}^+ \ \text{ wich } \ \text{implies}$$

$$\Im F_\mu(iy) = y + \eta(y) \ ,$$

where $\eta(y) = \int\limits_{-\infty}^{+\infty} \frac{y(1+t^2)}{y^2+t^2}\, \mathrm{d}\sigma(t)$. By Lemma 2.4, $F_\mu(iy) = iy - M_y + o(1)$ wich implies $\Im F_\mu(iy) = y$. Then, through identicaly results that $\eta(y) = o(1)$ as $y \to \infty$.

Observe that, for $|t| \geq y > 0$, $\frac{yt^2}{y^2+t^2} \geq \frac{1}{2} y$, then, $\eta(y) \geq \int\limits_{-\infty}^{+\infty} \frac{yt^2}{y^2+t^2}\, \mathrm{d}\sigma(t) \geq \int\limits_{\Delta_y} \frac{yt^2}{y^2+t^2}\, \mathrm{d}\sigma(t) \geq \frac{1}{2} y \sigma(\Delta_y)$. Therefore $y\sigma(\Delta_y) = o(1)$ as $y \to \infty$.

Again by the Nevanlinna representation we get $\frac{\mathrm{d}}{\mathrm{d}z} F_\mu(z) = 1 + \int\limits_{-\infty}^{+\infty} \frac{1+t^2}{(z-t)^2} \mathrm{d}\sigma(t)$.

For $z = x + iy \in \Gamma_{1/4}$ wich implies $|x| < \frac{y}{4}$ and

$$\begin{aligned} \left| \int\limits_{-\infty}^{+\infty} \frac{1+t^2}{(z-t)^2}\, \mathrm{d}\sigma(t) \right| &\leq \int\limits_{-\infty}^{+\infty} \frac{1+t^2}{|z-t|}\, \mathrm{d}\sigma(t) \leq \int\limits_{-\infty}^{+\infty} \frac{1+t^2}{(x-t)^2+y^2}\, \mathrm{d}\sigma(t) \\ &\leq \int\limits_{-\infty}^{+\infty} \frac{1+t^2}{t^2+y^2-\frac{|t|y}{2}}\, \mathrm{d}\sigma(t) \leq 2 \int\limits_{-\infty}^{+\infty} \frac{1+t^2}{t^2+y^2}\, \mathrm{d}\sigma(t) \ , \end{aligned}$$

since here use the inequality $t^2 + y^2 - \frac{|t|y}{2} \geq \frac{t^2+y^2}{2}$ for all t.

Furthermore, notice that $\int\limits_{I_{\sqrt{y}}} \frac{1+t^2}{t^2+y^2}\, \mathrm{d}\sigma(t) \leq \frac{1+y}{y+y^2}\, \sigma(\Delta_{\sqrt{y}}) = \frac{1}{y}\, \sigma(\Delta_{\sqrt{y}}) = \frac{1}{\sqrt{y}} \cdot \frac{1}{\sqrt{y}}\, \sigma(\Delta_{\sqrt{y}}) = \frac{1}{\sqrt{y}} \cdot \frac{\sqrt{y}\sigma(\Delta_{\sqrt{y}})}{y} = \frac{1}{\sqrt{y}}\, o(1)$. Here we used the fact that the function $t \mapsto \frac{1+t^2}{t^2+y^2}$ is increasing for $t > 0$ and $y > 1$.

On the other side, $\int\limits_{\Delta_{\sqrt{y}}} \underbrace{\frac{1+t^2}{t^2+y^2}}_{\leq 1}\, \mathrm{d}\sigma(t) \leq \sigma(\Delta_{\sqrt{y}}) = \frac{1}{\sqrt{y}}\, o(1)$. Since $y \leq |z| < \frac{\sqrt{17}}{4}$, for $|z| \in \Gamma_{1/4}$, we get the desired estimate.

Lemma 2.6. *Let μ be a probability measure on $\mathbb{R}$ such that $\lim\limits_{y\to\infty} y\mu(\Delta_y) = 0$. Then $\psi_\mu(iy) = M_y + o(1)$ as $y \to \infty$.*

Proof. By Lemma 2.4, $F_\mu(iy) = iy - M_y + h(y)$ with $\lim\limits_{y\to\infty} h(y) = 0$, while by Lemma 2.3(ii), $M_y = o(\ln y)$ if $y \to \infty$, follows that $F_\mu(iy) \in \Gamma_{1/4}$ for y large enough.
Thus for y large enough, $iy = F_\mu^{-1}(F_\mu(iy)) = F_\mu^{-1}(iy - M_y + h(y))$. However, let $z = iy - M_y + h(y)$, then $F_\mu^{-1}(z) = z + M_y - h(y) = z + o(|z|)$ as $|z| \to \infty$ and $z \in \Gamma_{1/4}$.
By Lemma 2.5 and $F_\mu^{-1}(z) = z + o(|z|)$, we have that $\frac{\mathrm{d}}{\mathrm{d}z} F_\mu^{-1}(z) = 1 + \frac{1}{\sqrt{|z|}}\, k(z)$ with $k(z) = o(1)$ as $|z| \to \infty$ and $z \in \Gamma_{1/4}$. Therefore

$$
\begin{aligned}
F_\mu^{-1}(iy - M_y + h(y)) - F_\mu^{-1}(iy) &= (iy - M_y + h(y) - iy)\frac{\mathrm{d}}{\mathrm{d}z} F_\mu^{-1}(z)|_{z=\gamma} \\
&= [-M_y + h(y)]\left(1 + \frac{k(\gamma)}{\sqrt{|\gamma|}}\right) \\
&= (-M_y + h(y))(1 + o(1)) \text{ as } y \to \infty\ .
\end{aligned}
$$

We get that

$$
\begin{aligned}
\psi_\mu(iy) &= F_\mu^{-1}(iy) - iy \\
&= \underbrace{F_\mu^{-1}(iy - M_y + h(y))}_{iy} + M_y - h(y) + M_y \cdot o(1) - h(y) \cdot o(1) - iy \\
&= M_y + o(1) \text{ as } y \to \infty\ .
\end{aligned}
$$

Proof of the Theorem 2.1

Proof. (i)⇒(ii): assume that μ satisfies condition (i) of the theorem. The Nevanlinna representation of $F_\mu(z)$ implies for $y>0$ that,

$$F_\mu(iy) = a + iy + \int_{-\infty}^{+\infty} \frac{1+iyz}{t-iy}\, \mathrm{d}\sigma(t)$$

wich is equivalent with

$$F_\mu(iy) = a + iy + \int_{-\infty}^{+\infty} \frac{t - y^2 t + i(yt^2+y)}{t^2+y^2}\, \mathrm{d}\sigma(t)\ ,$$

which means

$$\Im F_\mu(iy) = y + \eta(y),$$

$$\Re F_\mu(iy) = a + \xi(y),$$

with $\eta(y) = \int_{-\infty}^{+\infty} \frac{y(1+t^2)}{t^2+y^2}\, \mathrm{d}\sigma(t)$ and $\xi(y) = \int_{-\infty}^{+\infty} \frac{t(1-y^2)}{t^2+y^2}\, \mathrm{d}\sigma(t)$.
By Lemmas 2.2 and 2.4, $F_\mu(iy) = iy - M_y + o(1)$, $\Im F_\mu(iy) = y$, we have that $\eta(y) = o(1)$ if $y \to \infty$, and the same argument used in Lemma 2.5, $y\sigma(\Delta_y) = o(1)$ as $y \to \infty$. This estimate along with Lemma 2.3(i) allows us to conclude that $\xi(y) = o(\sqrt{y})$ as $y \to \infty$. Indeed

$$\begin{aligned}\frac{|\xi(y)|}{\sqrt{y}} &\le \frac{1}{\sqrt{y}} \cdot \frac{y\sqrt{y}}{y\sqrt{y}} \int_{-\infty}^{+\infty} \frac{|t(1-y^2)|}{t^2+y^2}\, \mathrm{d}\sigma(t) = y\sqrt{y} \int_{-\infty}^{+\infty} \frac{|t|\,|1-y^2|}{y^2(t^2+y^2)}\, \mathrm{d}\sigma(t) \\ &\le y\sqrt{y} \int_{-\infty}^{+\infty} \frac{|t|}{t^2+y^2}\, \mathrm{d}\sigma(t) = o(1)\ ,\end{aligned}$$

Here we used the inequality $\frac{|1-y^2|}{y^2} < 1$ for $y > 1$.

We can now estimate the imaginary part of $G_\mu(iy)$. Is know the fact that $\Im \frac{1}{z} = -\frac{\Im z}{|z|^2}$. Then

$$
\begin{aligned}
\Im G_\mu(iy) &= \Im \frac{1}{F_\mu(iy)} = -\frac{\Im F_\mu(iy)}{|F_\mu(iy)|^2} = \frac{-y+o(1)}{\Re^2 F_\mu(iy) + \Im^2 F_\mu(iy)} \\
&= \frac{-y+o(1)}{(a+o(\sqrt{y}))^2 + (y+o(1))^2} = \frac{-y+o(1)}{a^2+y^2} \\
&< \frac{-y+o(1)}{y^2} = -\frac{1}{y} + \frac{1}{y^2}\, o(1)\,.
\end{aligned} \tag{2.6}
$$

By Lemma 2.4, $\Im G_\mu(iy) = -\frac{1}{y} + \frac{1}{y} \int\limits_{-\infty}^{+\infty} \frac{t^2}{y^2+t^2}\, \mathrm{d}\mu(t)$ wich implies

$$
\int\limits_{-\infty}^{+\infty} \frac{yt^2}{y^2+t^2}\, \mathrm{d}\mu(t) = o(1).
$$

By proof of Lemma 2.5 results that

$$
\int\limits_{-\infty}^{+\infty} \frac{yt^2}{y^2+t^2}\, \mathrm{d}\mu(t) \geq \int\limits_{\Delta_y} \frac{yt^2}{y^2+t^2}\, \mathrm{d}\mu(t) \geq \frac{1}{2} y\mu(\Delta_y)\,,
$$

which lead at $y\mu(\Delta_y) = o(1)$ as $y \to \infty$ that is (ii).

(ii)$\Rightarrow$(i): suppose that μ satisfies condition (ii) of the theorem. We have $\psi_{\nu_n}(z) = \psi_\mu(nz) - M_n$ where the ν_n are defined as in condition (i) of the theorem. Notice that the functions ψ_{ν_n} are defined on a certain truncated cone $\Gamma_{\alpha,\beta}$ for every n.

By Lemma 2.6 for every fixed $y > \beta$, $\psi_{\nu_n}(iy) = \psi_\mu(iny) - M_n = M_{ny} - M_n + o(1)$, if $n \to \infty$ and $y \to \infty$.

Assuming without loss of generality that $\beta > 1$, the argument used in

Lemma 2.3, gives the following estimate

$$\begin{aligned}
|M_{ny} - M_n| &= \left| \int\limits_{I_{ny}} t \, \mathrm{d}\mu(t) - \int\limits_{I_n} t \, \mathrm{d}\mu(t) \right| \\
&\leq \int\limits_{I_{ny}-I_n} |t| \, \mathrm{d}\mu(t) = -ny\mu(\Delta_{ny}) - n\mu(\Delta_n) + \int\limits_n^{ny} \mu(\Delta_t) \, \mathrm{d}t \\
&\leq -ny\mu(\Delta_{ny}) - n\mu(\Delta_n) + \sup_{t\in[n,ny]} t\mu(\Delta_t) \int\limits_n^{ny} \frac{1}{t} \, \mathrm{d}t \\
&\leq -ny\mu(\Delta_{ny}) - n\mu(\Delta_n) + \sup_{t\in[n,ny]} t\mu(\Delta_t) \ln y = o(1) \,.
\end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} \psi_{\nu_n}(iy) = 0$.

Moreover

$$\limsup_{y\to\infty} \left| \frac{\psi_{\nu_n}(iy)}{y} \right| \leq \limsup_{y\to\infty} \frac{-2k + k \ln y}{y} =$$

$$\limsup_{y\to\infty} \frac{-2k}{y} + k \limsup_{y\to\infty} \frac{\ln y}{y} = 0 \,.$$

Hence Proposition 2.1(c), ν_n converges weakly to a measure ν and $\psi_\nu(iy) = 0$ for every $y > \beta$. The identicaly theorem then implies that $\psi_\nu(z) = 0$, for every that $z \in \Gamma_{\alpha,\beta}$ which implies that $\nu = \delta_0$.

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