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The approximation of a Polynomial's Measure, with Applications towards Jensen's Theorem[1]

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Abstract

We introduce the notion of measure of a polynomial $F(z)$ with complex coefficients, then we give an interpretation for it as an integral, using Jensen's theorem. By introducing a new polynomial F must be evaluated, depending on the measure of the new polynomial, according only to the expression of F, or to other integral expressions.

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1 Introduction

For $F(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, F(x) \in \mathbb{C}[x], a_0 \neq 0, a_n \neq 0$, $n \geq 1$ with the roots $x_1, x_2, \ldots, x_n \in \mathbb{C}$; repeated according to their multiplicity, then we introduce the notion of measure of polynomial $F(z)$ with complex coefficients.

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$$M(F) = M[F(x)] = |a_n| \cdot \prod_{j=1}^{n} max\{1, |x_j|\}$$

and there will be found an interpretation for it as an integral, by using Jensen's theorem:

$$ln[M(f)] = \frac{1}{2\pi} \int_0^{2\pi} ln|F(e^{i\theta})|d\theta.$$

We can determinate effectively the series $(F_m)_{m\geq 1}$ and then, $M(F)$ which is a real number, from relations:

$$2^{-n\cdot 2^{-m}} \cdot \|F_m\|^{2^{-m}} \leq M(F) \leq \|F_m\|^{2^{-m}}.$$

$$\lim_{m\to\infty} \|F_m\|^{2^{-m}} = M(F).$$

Now by anew polynomial which depends on F, and that has all the roots in $D(0,1)$, the degree of the polynomials F must be evaluated, depending either on the edge of all the real roots $R = 1 + \sum_{k=0}^{n-1} |\frac{a_k}{a_n}| = \frac{L(F)}{|a_n|}$, on the maximum and minimum of $F(z)$ when $|z| = 1$, or on the measure of a new polynomial or, least but not last, depending only the expression of F.
For example:

$$\frac{\ln\{\min\limits_{|z|=R} |F(z)|\} - \ln|a_n|}{ln|L(F)| - ln|a_n|} \leq n \leq \frac{\ln\{\max\limits_{|z|=R} |F(z)|\} - ln|a_n|}{ln|L(F)| - ln|a_n|},$$

$$n = \frac{\frac{1}{2\pi}\int_0^{2\pi} \ln|F[\frac{L(F)}{a_n} \cdot e^{i\theta}]|d\theta - \ln|a_n|}{\ln[L(F)] - \ln|a_n|}.$$

Also, other relation allowing the evaluation of maximum and minimum for $F(z)$ when $|z| = R$ has been deducted; depending on the edge R, and depending on the polynomial's coefficients and degree. Then we can compare the previous results with some basic results on Complex Analysis for $R = \frac{L(P)}{|a_n|}$ the edge of all the real roots, such as 'The Cauchy's integral formula for polynomials on R'-radius circles and 'Maximum Principle'.

2 Measure of a polynomial

Definition 2.1. *Let* $F(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, n \geq 1,$ $a_n \neq 0$, $F(x) \in \mathbb{C}[x]$, *with the roots* $x_1, x_2, \ldots, x_n \in \mathbb{C}$; *repeated according to their multiplicity, then by definition, the measure of the polynomials* F, *noted by* $M(F)$, *is:*

$$M(F) = M[F(x)] = |a_n| \cdot \prod_{j=1}^{n} \max\{1, |x_j|\}$$

Definition 2.2. *Let* $F(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, n \geq 1, a_n \neq 0,$ *Then the norm of the polynomial* F, *noted by* $\|F\|$, *will be:*

$$\|F\| = \sqrt{a_0^2 + a_1^2 + a_2^2 + \ldots + a_n^2}$$

And the length of the polynomial F *noted by* $L(F)$ *is:*

$$L(F) = \sum_{k=0}^{n} |a_k|.$$

Theorem 2.1. *For* $F(x) \in \mathbb{C}[x]; F(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, n \geq 1, a_n \neq 0$,*with the toots* $x_1, x_2, \ldots, x_n \in \mathbb{C}$; *repeated according to their multiplicity, then we have:*

a) $M(F) = \dfrac{|a_0|}{\prod_{j=1}^{n} \min\{1, |x_j|\}}$; *b)* $M[x^n \cdot F(\frac{1}{x})] = M[F(x)]$;
c)$M(P \cdot Q) = M(P) \cdot M(Q)$, *for all* $P, Q \in \mathbb{C}[x]$; *d)* $M[F(x^k)] = M[F(x)]$
e) $M^2(F) + |a_o a_n|^2 \cdot M^{-2}(F) \leq \|F\|^2$.

Proof. a) From Viete's formulas we have $\prod_{j=1}^{n} |x_j| = \left|\dfrac{a_0}{a_n}\right|$.

But, $\prod_{j=1}^{n}|x_j| = \prod_{j=1}^{n}\max\{1,|x_j|\}\cdot\prod_{j=1}^{n}\min\{1,|x_j|\} = \left|\frac{a_0}{a_n}\right|$.

where we have $|a_n|\cdot\prod_{j=1}^{n}\max\{1,|x_j|\} = \frac{|a_0|}{\prod_{j=1}^{n}\min\{1,|x_j|\}}$.

b) $H(x) = x^n\cdot F(\frac{1}{x}) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$,

$M\left[x^n\cdot F\left(\frac{1}{x}\right)\right] = a_0\cdot\prod_{j=1}^{n}\max\{1,|y_j|\}$ where $H(j_j) = 0$ and $y_j = \frac{1}{x_j}$, $j = \overline{1,n}$.

Now $M\left[x^n\cdot F\left(\frac{1}{x}\right)\right] = \frac{|a_0|}{\prod_{j=1}^{n}\min\{1,|x_j|\}}$ and from a) is obvious.

c) and d) are easy to prove, also see [1] from References.

e) see[1] or[2] from References.

f)From (e) we have that: $M^2(F)\leq\|F\|^2$ and $M(F)\leq\|F\|$

Theorem 2.2. *If* $f(x) = a_n\cdot\prod_{j=1}^{n}(x-x_j), a_n\neq o$, *where* $x_j\in\mathbb{C}; j = \overline{1,n}, n\geq 1$ *are the polynomials roots, repeated according to their multiplicity, and the polynomials*

$F_m(x) = \pm a_n^{2m}\cdot\prod_{j=1}^{n}(x - x_j^{2^m}); m\geq 0$.

Then we can calculate $F_m(x)$ *according to Graeffe's Method, that is:*

i) $F_0(x) = F(x)$

ii) Then for $m = \overline{0,n-1}$, *we can find* $\{G_m(x), H_m(x), F_{m+1}(x)\}\in C[x]$ *in order to have:* $F_m(x) = G_m(x^2) - x\cdot H_m^2(x)$. ,$F_{m+1}(x) = G_m^2(x) - x\cdot H_m^2(x)$

iii) Finally, we find $F_m(x)$ *for all* $m\geq 0$.

Proof. See [3] from References.

Theorem 2.3. *If $F(x) \in C[x]$; $gradF \geq 1$ and F_m, $m \geq 0$, the polynomial series associated through the Graeffe's Method, them:*

$$2^{-n \cdot 2-m} \cdot \|F_m\|^{2^{-m}} \neq \|F_m\|^{2^{-m}}$$

and

$$\lim_{n\to\infty} \|F_m\|^{2^{-m}} = M(F).$$

Proof. See [3] from References.

Remark 2.1. *This theorem allows evaluating the polynomial's measure as many as exact decimals.*

3 Jensen's Equality and its Applications

Theorem 3.1. Jensen's equality. *Let $F(x)$ an analytic function in a region which contains the closed disk $\overline{D}(0; R); R > 0$ in the complex plane, if $n \geq 1, x_1, x_2, \ldots, x_n \in C$, $|x_j| < R$, for all $i = \overline{1, n}$, are the zeros of F in the interior of $D(0; R)$ repeated according to their multiplicity and if $F(0) \neq 0$, then:*

$$\ln|F(0)| = -\sum_{j=1}^{n} \ln\left(\frac{R}{x_j}\right) + \frac{1}{2\pi}\int_0^{2\pi} \ln|F(R \cdot e^{i\theta})|d\theta \ \ or \ :$$

$$n \ln R = \frac{1}{2\pi}\int_0^{2\pi} \ln|F(R \cdot e^{i\theta})|d\theta - \ln|F(0)| + \ln\left(\prod_{j=1}^{n}|x_j|\right).$$

Proof. See [2] or [5] from References.

Remark 3.1. *This formula establishes a connection between the moduli of the zeros of the function F inside the disk $|z| < R$ and the values of $|F(z)|$ on the circle $|z| = R$, and can be seen as a generalization of the mean value property of harmonic functions.*

Consequence 3.1. *Let $F(x)$ an analytic function in a region which contains the closed disk $\overline{D}(0;1)$ in the complex plane, $m \geq 1$ is the number of all zeros of F in the interior of $D(0;1)$ repeated according to multiplicity, then:*

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \ln|F(e^{i\theta})|d\theta - \ln|F(0)| + \ln\left(\prod_{j=1}^{n}|x_i|\right).$$

Proof. Is obvious by Theorem 3.1 for $R = 1$.

Theorem 3.2. *If $F(x) = a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0, F(x) \in C[x], a_0 \neq 0, a_m \neq 0, m \geq 1$ then:*

$$\ln[M(F)] = \frac{1}{2\pi}\int_0^{2\pi} \ln|F(e^{i\theta})|d\theta$$

and:

$$\min_{|z|=1}\{|F(z)|\} \leq M(F) \leq \max_{|z|=1}\{|F(z)|\}$$

Proof. Let be: $x_1, x_2, \ldots, x_m \in C$, for all $i = \overline{1,m}$ all the roots, repeated according to multiplicity of $F(x)$ in the complex plane; and $x-1, x_2, \ldots x_n$ the roots repeated according to multiplicity of $F(x)$ in the interior of $D(0;1)|x-j| < 1; j + \overline{1,n}$.
Because $F(x)$ is a polynomial analytic function and $F(0) = a_0 \neq 0$ we are within the hypothesis of Consequence 3.1, and we have:

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \ln|F(e^{i\theta})|d\theta - \ln|F(0)| + \ln\left(\prod_{j=1}^{n}|x_j|\right).$$

that is:

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|F(e^{i\theta})|d\theta = ln|F(0)| - \ln\left(\prod_{j=1}^{n}|x_j|\right).$$

In conclusion,

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|F(e^{i\theta})|d\theta = \ln\frac{|a_0|}{\prod_{j=1}^{n}|x_j|}.$$

But $\prod_{j=1}^{n}|x_j| = \prod_{j=1}^{m}\min\{1,|x_j|\}$ and according to Theorem 2.1 a), we have:

$$\frac{1}{2\pi}\int_0^{2\pi}\ln|F(e^{i\theta})|d\theta = \ln(M(F)).$$

Next, by maximizing and minimizing, we obtain:

$$\frac{1}{2\pi}\int_0^{2\pi}\ln[\min_{|z|=1}(|F(z)|)]d\theta \leq \ln(M(F)) \leq \frac{1}{2\pi}\int_0^{2\pi}\ln[\max_{|z|=1}(|F(z)|)]d\theta.$$

that is:

$$\min_{|z|=1}\{|F(z)|\} \leq M(F) \leq \max_{|z|=1}\{|F(z)|\}.$$

Theorem 3.3. *If* $F(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0, a_i \in C; i = \overline{1,n}, n \geq 1,$ *and* $a_0 \neq 0, a_n \neq 0$ *if* $x_1, x_2, \ldots, x_n$ *are the roots repeated according to multiplicity of* $F(x), F(x_j) = 0, = \overline{1,n}$ *then exists* $R > 0$*;*
$R > \max\left\{1, \sum_{k=0}^{n-1}|\frac{a_k}{a_n}|\right\}$ *or* $R > 1 + \max_{0\leq k<n}\{|\frac{a_k}{a_n}|\}$ *or simply* $R = 1\sum_{k=0}^{n-1}|\frac{a_k}{a_n}|$
$= \frac{L(F)}{|a_n|}$*, with* $|x_i| < R; i = \overline{1,n}.$

Proof. See [1] or [4] from References.

Remark 3.2. *We shall now give a few inequalities which resulted by combining the condition from Theorem 3.3., so that all the roots of a polynomial* $F(x)$ *to be, within the moduli in interval* $(0,R)$*, and the Jensen's equality. We have inferred these inequalities by using Theorem 3.2. where the author introduced the measure of a polynomial in the above equality.*

Theorem 3.4. *If* $F(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0, a_i \in C; i = \overline{1,n}, n \geq 1$ *and* $R \in R, R > 0,$ *so that* $F(x_i) = 0, |x_i| < R; i = \overline{1,n},$ *where* $x-1, x-2,$ *dots*$, x_n$ *are all the roots repeated according to multiplicity of* $F(x)$ *and* $a_0 \neq, a_n \neq 0,$ *then:*
a) If $G(y) = F(R\cdot y)$ *and* $G(y_i) = 0, i = \overline{1,n},$ *that is* $y_1, y_2, \ldots, y_n$ *are the roots repeated according to multiplicity of* $G(y)$*, then* $y_i = \frac{x_i}{R}$ *and* $|y_i| < 1.$

b)Exists $\epsilon \in R; 0 < \epsilon < 1$ *so that* $|a_0| \cdot (1-\epsilon)^n \leq \max\limits_{|z|=R(1-\epsilon)}\{|F(z)|\}$

and

c) $\frac{\ln\{\min\limits_{|z|=R}|F(z)|\} - \ln|a_n|}{\ln R} \leq n \leq \frac{\ln\{\max\limits_{|z|=R}|F(z)|\} - \ln|a_n|}{\ln R}$.

Proof. a) Let $y_i = \frac{x_i}{R}; i = \overline{1,n}$, then $G(y_i) = F(R \cdot y_i) = F(R \cdot \frac{x_i}{R} = F(x_i) = 0)$,for all $i = \overline{1,n}$.

In conclusion, $y_i = \frac{x_i}{R}$ are the zeros of $G(y)$.

Moreover, $|y_i| = |x_y|R < \frac{R}{R} = 1$, that involves $|y_i| < 1$; $y = \overline{1,n}$.

b)Because $G(y) = F(R \cdot y)$, we have:

$G(y) = a_n R^n \cdot y^n + a_{n-1}R^{n-1} \cdot y^{n-1} + \ldots + a_1 R \cdot y + a_0, a_i \in \mathbb{C}; i = \overline{1,n}$

We choose $\epsilon > 0$ so that $R_1 = 1 - \epsilon$ and $|y_i| < 1 - \epsilon < 1, i = \overline{1,n}$.

This choise was possible due to the fact that $|y_i| < 1, i = \overline{1,n}$.

From Jensen's equality for $G(y)$, we have:

$$n \ln R_1 = \frac{1}{2\pi}\int_0^{2\pi} \ln|G(R_1 \cdot e^{i\theta})|d\theta - \ln|G(0)| + \ln\left(\prod_{j=1}^{n}|y_j|\right)$$

Because for $j = \overline{1,n}, |y_i| < R_1 < 1$, we have $0 \leq \prod\limits_{j=1}^{n}|y_j| < 1$ which implies $\ln\left(\prod\limits_{j=1}^{n}|y_j|\right) < 0$.

And next $G(0) = a_0 \neq 0$ and $\int_0^{2\pi} \ln|G(R_1 \cdot e^{i\theta)}|d\theta < \int_o^{2\pi} \ln[\max\limits_{|z|=R_1}|G(z)|]d\theta$,

Therefore, from the previous relation we have:

$$n \ln R_1 \leq \frac{1}{2\pi} \cdot 2\pi \cdot \ln[\max_{|z|=R_1}|G(z)|] - ln|a_0|$$

and because $R_1 = 1 - \epsilon, \epsilon > 0$, that is:

$$\ln(1-\epsilon)^n \leq \ln\frac{\max\limits_{|z|=1-\epsilon}|G(z)|}{|a_0|}$$

or,

$$(1-\epsilon)^n \leq \frac{\max\limits_{|z|=1-\epsilon}|G(z)|}{|a_0|}. \tag{3.1}$$

As $\max\limits_{|z|=1-\epsilon}|G(z)| = \max\limits_{|z|=1-\epsilon}|F(R\cdot z)|$, by nothing $R\cdot z = z_1$, implies that $z = \frac{z_1}{R}$, and then:

$$\max_{|z|=1-\epsilon}|G(z)| = \max_{|\frac{z_1}{R}|=1-\epsilon}|F(z_1)| = \max_{|z_1|=R(1-\epsilon)}|F(z_1)|.$$

As a result:

$$\max_{|z|=R_1=1-\epsilon}|G(z)| = \max_{|z|=R(1-\epsilon)}|F(z_1)| = \max_{|z|=R(1-\epsilon)}|F(z)|. \tag{3.2}$$

From relation (3.1) and relation (3.2) we have:

$$(1-\epsilon)^n \leq \frac{\max\limits_{|z|=R(1-\epsilon)}|F(z)|}{|a_0|}$$

which is equivalent with

$$|a_0|\cdot(1-\epsilon)^n \leq \max_{|z|=R(1-\epsilon)}\{|F(z)|\},$$

with $\epsilon > 0$, conveniently chosen.

c) By applying Theorem 3.2. to the polynomial $G(z) = F(R\cdot z)$, is obtained, due to the fact that $|y_i| < 1; i = \overline{1,n}$:

$$\min_{|z|=1}\{G(z)\} \leq M[G(z)] \leq \max_{|z|=1}\{G(z)\},$$

that is:

$$\min_{|z|=1}\{|F(R\cdot z)|\} \leq M[G(z)] \leq \max_{|z|=1}\{|F(R\cdot z)|\}.$$

By nothing $R\cdot z = z_1$, we obtain:
$|z| = 1$ is equivalent to $|z_1| = R$; $\min\limits_{|z|=1}\{|F(R\cdot z)|\} = \min\limits_{|z_1|=R}\{F(z_1)\}$
$\max\limits_{|z|=1}\{|F(z_1)|\} = \max\limits_{|z_1|=1}\{|F(z_1)|\}$.

Then:

$$\min_{|z_1|=R}\{|F(z_1)|\} \leq M[F(z_1)] \leq M[G(z)] \leq \max_{|z_1|=R}\{F(z_1)\}.$$

And afterwards nothing $z_1 = z$ we have:

$$\min_{|z|=R}\{|F(z)|\} \leq M[G(z)] \leq \max_{|z|=R}\{F(z)\}. \tag{3.3}$$

Still, $M[F(R\cdot z)] = M[G(z)] = M[a_nR^nz^n + a_{n-1}R^{n-1}y^{n-1} + \ldots + a_1Ry + a_0]$.
For $G(z) = a_nR^n \cdot \prod_{j=1}^{n}(z - y_j)$ we have from definition:
$M[G(z)] = |a_nR^n| \cdot \prod_{j=1}^{n} \max\{1, |y_j|\}$, that is: $M[G(z)] = M[F(R\cdot z)] =$
$= |a_nR^n| \prod_{j=1}^{n} \max\{1, |y_j|\}$.
But due to the fact that $|y_j| < 1, j = \overline{1,n}$, we obtain

$$M[F(R\cdot z)] = M[G(z)] = |a_n| \cdot R^n \tag{3.4}$$

Therefore, from relation (3.3) an (3.4), we obtain:

$$\min_{|z|=R}\{|F(z)|\} \leq M[G(z)] = |a_n| \cdot R^n \leq \max_{|z|=R}\{F(z)\}.$$

And that implies: $\ln[\min_{|z|=R}\{F(z)\}] \leq \ln[|a_n| \cdot R^n] \leq \ln[\max_{|z|=R}\{|F(z)|\}]$ which is equivalent with

$$\ln[\min_{|z|=R}\{|F(z)|\}] - \ln|a_n| \leq n\ln R \leq \ln[\max_{|z|=R}\{|F(z)|\}] - \ln|a_n|,$$

and because $R > 1$ we have:

$$\frac{\ln\{\min_{|z|=R}|F(z)|\} - \ln|a_n|}{\ln R} \leq n \leq \frac{\ln\{\max_{|z|=R}|F(z)|\} - \ln|a_n|}{\ln R}.$$

Consequence 3.2. *If $F(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0, a_i \in \mathbb{C}; i = \overline{1,n}, n \geq 1, F(x_i) = 0, i = \overline{1,n}$, where $x_1, x_2, \ldots, x_n,$, are the roots repeated according to multiplicity of $F(x)$ and $a_0 \neq 0, a_n \neq 0$, then:*

a)

$$\frac{\ln\{\min\limits_{|z|=\frac{L(F)}{a_n}}|F(z)|\}-\ln|a_n|}{\ln L(F)-\ln|a_n|}\leq n\leq\frac{\ln\{\max\limits_{|z|=\frac{L(F)}{|a_n|}}|F(z)|\}-\ln|a_n|}{\ln L(F)-\ln|a_n|};$$

b) *Moreover, if* $L(F)=1$ *then:*

$$\frac{\ln\{\min\limits_{|z|=\frac{1}{a_n}}|F(z)|\}-\ln|a_n|}{\ln|a_n|}\geq n\geq\frac{\ln\{\max\limits_{|z|=\frac{1}{|a_n|}}|F(z)|\}-\ln|a_n|}{\ln|a_n|};$$

c) *Furthermore, if* $a_n=1$ *then:* $\frac{\ln\{\min\limits_{|z|=L(F)}|F(z)|\}}{\ln L(F)}\leq n\leq\frac{\ln\{\max\limits_{|z|=L(F)}|F(z)|\}}{\ln L(F)}$.

Proof. a) We can prove it by using Theorem 3.3 for $R=\frac{L(f)}{|a_n|}$ and from Theorem 3.4. c)
b),c) is obvious from a).

Theorem 3.5. The Maximum Principle. *An analytic function on an open set* $U\subset\mathbb{C}$ *assumes its maximum modulus on the boundary. Moreover, if* f *is analytic and takes at least two distinct values on an open connected set* $U\subset\mathbb{C}$, *then*

$$|F(z)|<\sup_{z\in U}|F(z)|, z\in U$$

Proof. See [5] to References.

Remark 3.3. *If* $F(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$, $a_1\in\mathbb{C}$; $i=\overline{1,n}$, $n\geq 1$, $a_n\neq 0$, $R>0$, *then:*
a) $|F(z)|\leq\max\limits_{|z|=R}\{F(z)\}\leq|a_n|R^n+|a_{n-1}|R^{n-1}+\ldots+|a_1|R+a_0$, *for each* $z\in D(0;R)\subset\mathbb{C}$.
b) If $R\in R, R>0$, *so that* $F(x_i)=0$, $|x_i|<R; i=\overline{1,n}$, *where* $x_1,x_2,\ldots,x_n$ *are the roots repeated according to multiplicity of* $F(x)$ *and* $a_0\neq 0$, $a_n\neq 0$, *then:*
$|a_n|R^n\leq\max\limits_{|z|=R}\{|F(z)|\}\leq|a_n|R^n+|a_n-1|R^{n-1}+\ldots+|a_1|R+|a_o|$, *for each* $z\in D(0;R)\subset\mathbb{C}$.

Proof. a) we are in conditions of the **Maximum Principle**. For $U = D(0;R) \subset \mathbb{C}$:
$|F(z)| \leq \max_{|z|=R}\{|F(z)|\}$, for each $z \in D(0;R) \subset \mathbb{C}$.
But
$\max\limits_{|z=R|}\{|F(z)|\} = \max\limits_{|z|=R} |a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0| \leq |a_{n-1}|R^{n-1} + \ldots + |a_1|R + |a_0| = a_n R^n + a_{n-1}R^{n-1} + \ldots + a_1 R + a_0$.
So we have the result
b) From Theorem 3.4 c) we have $|a_n|R^n \leq \max\limits_{|z|=R}\{|F(z)|\}$ and from a) of this Theorem we have:
$|a_n|R^n \leq \max\limits_{|z|=R}\{|F(z)|\} \leq |a_n|R^n + |a_{n-1}|R^{n-1} + \ldots + |a_1|R + a_0$ for each $z \in D(0;R) \subset \mathbb{C}$.

Theorem 3.6. *If $F(x) + a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, a_i \in \mathbb{C}$; $i = \overline{1,n}, n \geq 1$ and $R = \frac{L(F)}{|a_n|}, R > 1$ where $x_1, x_2, \ldots, x_n$ are the roots repeated according to multiplicity of $F(x)$ and $a_0 \neq 0, a_n \neq 0$, then:*

$$n = \frac{\frac{1}{2\pi}\int_0^{2\pi} \ln\left|F\left[\frac{L(F)}{|a_n|}\cdot e^{i\theta}\right]\right| d\theta - \ln|a_n|}{\ln\left[L(F)\right] - \ln|a_n|}.$$

Proof. We are in condition of Theorem 3.2. and we have for $G(z) = F(R \cdot z)$

$$\ln[M(G)] = \frac{1}{2\pi}\int_0^{2\pi} \ln|G(e^{i\theta})|d\theta.$$

Also we have $M[G(z)] = |a_n|R^n$ from the Theorem 3.4.c)
From these relation, we have: $\ln(|a_n|R^n) = \frac{1}{2\pi}\int_0^{2\pi} \ln|G(e^{i\theta})|d\theta$
where

$$n = \frac{\frac{1}{2\pi}\int_0^{2\pi} \ln|G(e^{i\theta})|d\theta - \ln|a_n|}{\ln(R)}.$$

Now we can take from hypothesis $R = 1 + \sum_{k=0}^{n-1}|\frac{a_k}{a_n}| = \frac{L(F)}{|a_n|}$ and from Theorem 3.3. $F(x_i) = 0, |x_i| < R$; $i = \overline{1,n}$.

Also because $G(e^{i\theta}) = F(R \cdot e^{i\theta})$ we obtain:

$$n = \frac{\frac{1}{2\pi}\int_0^{2\pi} \ln|F[\,\frac{L(F)}{|a_n|}\cdot e^{i\theta}\,]|\,d\theta - \ln|a_n|}{\ln\left[\frac{L(F)}{|a_n|}\right]}$$

or

$$n = \frac{\frac{1}{2\pi}\int_0^{2\pi} \ln\left|F\left[\frac{L(F)}{|a_n|}\cdot e^{i\theta}\right]\right|d\theta - \ln|a_n|}{\ln[L(F)] - \ln|a_n|}.$$

Theorem 3.7. The Cauchy's integral formula for polynomials on circles. *If* $F(z) = a_m\prod_{j=1}^{m}(x - x_j)$, *then the number of indices* j *for which* $|x_j| < R$ *is "n":*

$$n = \frac{1}{2\pi i}\int_{|z|=R}\frac{F'(z)}{F(z)}dz$$

provided no x_j *lies on* $Fr(D) : |z| = R$.

Proof. $\frac{1}{2\pi i}\int_{|z|=R}\frac{F'(z)}{F(z)}dz = \frac{1}{2\pi i}\cdot\int_{|z|=R}\sum_{j=0}^{m}\frac{1}{z - x_j}dz =$

$\frac{1}{2\pi i}\cdot\sum_{j=0}^{m}\int_{|z|=R}\frac{1}{z - x_j}dz = \frac{1}{2\pi i}\cdot\sum_{j=0}^{n}\int_{|z-x_j|=e}\frac{1}{z - x_j}dz = \frac{1}{2\pi i}\cdot\sum_{j=0}^{n}2\pi i = n$

See [5] to References.

Remark 3.4. *If* $F(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0, a_i \in \mathbb{C}$; $i = \overline{1,n}, n \geq$, *and* $R = \frac{L(F)}{|a_n|}, R > 1$, *where* $x_1, x_2, \ldots, x_n$ *are the roots repeated according to multiplicity of* $F(x)$ *and* $a_0 \neq 0, a_n \neq 0$, *then:*

$$n = \frac{1}{2\pi i}\int_{|z|=\frac{L(F)}{|a_n|}}\frac{F'(x)}{F(z)}dz = \frac{\frac{1}{2\pi}\int_0^{2\pi}\ln\left|F\left[\frac{L(F)}{|a_n|}\cdot e^{i\theta}\right]\right|d\theta - \ln|a_n|}{\ln[L(F)] - \ln|a_n|}$$

Proof. If we take $R = 1 + \sum_{k=0}^{n-1}\left|\frac{a_k}{a_n}\right| = \frac{L(F)}{|a_n|}$ for the "n" from previous theorem, what is now the number of all zeros, we can find first equality, the another was given in Theorem 3.6.

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