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Fractional Derivate of Riemann-Liouville via Laguerre Polynomials[1]

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Abstract

We show that well known properties of Laguerre polynomials permit to motivate the definition of Riemann-Liouville for the fractional derivative.

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1 Introduction

The associated Laguerre polynomials are given by [1]:

(1) $$L_n^k(x) = (-1)^k \frac{d^k}{dx^k} L_{n+k}(x),\ k = 0, 1, 2, \dots$$

(2) $$= \sum_{r=0}^{n} (-1)^r \binom{n+k}{n-r} \frac{x^r}{r!},$$

therefore

(3) $$L_n(y) = L_n^0(y) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{y^r}{r!},$$

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and thus

$$L_0(y) = 1,\ L_1(y) = 1 - y,\ L_2(y) = \frac{1}{2}(2 - 4y + y^2), \ldots \tag{4}$$

In this work we accept that (1) is valid for $k = -1, -2, \ldots$, with two implications:
1. It permits to obtain an expression for $L_n(x)$ in terms of the $L_m^k(x)$;
2. It motivates the fractional derivative of Riemann-Liouville [2-4], wich are studied in Section 2 and 3, respectively.

2 $L_n(x)$ in terms of their associated polynomials

We put $k = -N = -1, -2, \ldots$ in (1) and we apply the operator $\frac{d^N}{dx^N}$ to deduce that:

$$L_{n-N}(x) = (-1)^N \frac{d^N}{dx^n} L_n^{-N}(x),\ n \geq N, \tag{5}$$

but using (2) it is easy to show the property [5]:

$$L_q^{p-q}(y) = (-1)^{p-q} \frac{p!}{q!} y^{q-p} L_p^{q-p}(y), \tag{6}$$

then

$$L_n^{-N}(x) = (-1)^N \frac{(n-N)!}{n!} x^N L_{n-N}^N(x), \tag{7}$$

and thus (5) implies:

$$L_{n-N}(x) = \frac{(n-N)!}{n!} \frac{d^N}{dx^N}(x^N L_{n-N}^N),$$

$$= \frac{(n-N)!}{n!} \sum_{m=0}^{N} \begin{pmatrix} N \\ m \end{pmatrix} \frac{d^{N-m}}{dx^{N-m}} x^N \cdot \frac{d^m}{dx^m} L_{n-N}^N,$$

$$= \frac{N!(n-N)!}{n!} \sum_{m=0}^{N} (-1)^m \begin{pmatrix} N \\ m \end{pmatrix} \frac{x^m}{m!} L_{n-N-m}^{N+m} \tag{8}$$

where we have employed the known relation ([1]):

$$\frac{d^b z^a}{dz^b} = \frac{a!}{(a-b)!} z^{a-b}, \quad \frac{d^c}{dx^c} L_b^a = (-1)^c L_{b-c}^{a+c}. \tag{9}$$

In (8) the upper limit of Σ may be $(n-M)$ because $L_{n-N-m}^{N+m} = 0$ if $m > (n-N)$, then finnaly (8) adopts the form:

$$\begin{pmatrix} n \\ r \end{pmatrix} L_r(x) = \sum_{m=0}^{r} (-1)^m \begin{pmatrix} n-r \\ m \end{pmatrix} \frac{x^m}{m!} L_{r-m}^{n-r+m}(x) \tag{10}$$

which is not common in the literature. The expansion (10) permits to write $L_r(x)$ in terms of their associated polynomials.

3 Fractional derivative of Riemann-Liouville

From (1) and (7) it is clear that:

$$\frac{d^{-N}}{dx^{-N}} L_{n-N}(x) = \frac{(n-N)!}{n!} x^N L_{n-N}^N(x). \tag{11}$$

On the other hand, the definition (2) and the recurence relation ([1]):

$$L_q^{p-1} = L_q^p - L_{q-1}^p = \frac{d}{dx}(L_q^{p-1} - L_{q+1}^{p-1}) \tag{12}$$

imply the known integral property ([1]):

$$\int_0^x L_m(t) L_n(x-t) dt = \int_0^x L_{m+n}(t) dt = L_{m+n}(x) - L_{m+n+1}(x) \tag{13}$$

If in (13) we put $n = 0, 1, 2, \ldots$, and use (4) and the recurrence expression ([1]):

$$x L_q^{p+1} = (p+q+1) L_p^q - (q+1) L_{q+1}^p, \tag{14}$$

then it is immediate that:

$$\int_0^x L_{n-1}(t) dt = \frac{x}{n} L_{n-1}^1$$

$$\int_0^x (x-t)L_{n-2}(t)dt = \frac{x^2}{n(n-1)}L_{n-2}^2$$

$$\vdots \quad \vdots$$

$$\frac{1}{(N-1)!}\int_0^x (x-t)^{N-1}L_{n-N}(t)dt = \frac{(n-N)!}{n!}x^N L_{n-N}^N(x) \tag{15}$$

that in union of (11) implies the interesting relation:

$$\frac{d^{-N}}{dx^{-N}}L_{n-N}(x) = \frac{1}{(N-1)!}\int_0^x (x-t)^{N-1}L_{n-N}(t)dt, \tag{16}$$

which is a strong motivation for the fractional derivative of Riemann-Liouville ([2-4]):

$$\frac{d^q}{dx^q}f(x) = \frac{1}{\Gamma(-q)}\int_0^x \frac{f(t)}{(x-t)^{1+q}}dt, \; q < 0 \tag{17}$$

for the case $q = -N = -1, -2, \ldots$

The generalization of (11) and (16) is given by [1]:

$$\frac{d^{-\beta}}{dx^{-\beta}}[x^\alpha L_m^\alpha(x)] = \frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}t^\alpha L_m^\alpha(t)dt \; ,$$

(18)

$$= \frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha+\beta+m+1)}x^{\alpha+\beta}L_m^{\alpha+\beta(x)} \; ,$$

for $\alpha > -1$ and $\beta > 0$.

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