General Mathematics Vol. 16, No. 1 (2008), 5-9

A Note on Heredity for Terraced Matrices[1]

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In Memory of Myrt Naylor Rhaly (1917-2006)

Abstract

A terraced matrix M is a lower triangular infinite matrix with constant row segments. In this paper it is seen that when M is a bounded linear operator on ℓ^2, hyponormality, compactness, and noncompactness are inherited by the "immediate offspring" of M. It is also shown that the Cesàro matrix cannot be the immediate offspring of another hyponormal terraced matrix.

2000 Mathematical Subject Classification: 47B99

Key words: Cesàro matrix, terraced matrix, hyponormal operator, compact operator

1 Introduction

Assume that $\{a_n\}$ is a sequence of complex numbers such that the associated *terraced matrix* $M = \begin{pmatrix} a_0 & 0 & 0 & \dots \\ a_1 & a_1 & 0 & \dots \\ a_2 & a_2 & a_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ is a bounded linear operator on

[1] *Received 19 February, 2007*

Accepted for publication (in revised form) 4 December, 2007

ℓ^2; these matrices have been studied in [**2**] and [**3**]. We recall that M is said to be *hyponormal* on ℓ^2 if $\langle [M^*, M]f, f\rangle = \langle (M^*M - MM^*)f, f\rangle \geq 0$ for all f in ℓ^2. It seems natural to ask whether hyponormality is inherited by the terraced matrix arising from any subsequence $\{a_{n_k}\}$. To see that the answer is no, we consider the case where $M = C = \begin{pmatrix} 1 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$, the Cesàro matrix. In [**4**, Corollary 5.1] it is seen that the terraced matrix associated with the subsequence $\{\frac{1}{2n+1} : n = 0, 1, 2, ...\}$ is not hyponormal, although the Cesàro matrix itself is known to be a hyponormal operator on ℓ^2 (see [**1**]).

Consequently, we turn our attention to a more modest result and consider hereditary properties of the terraced matrix arising from one special subsequence; we will regard $M' = \begin{pmatrix} a_1 & 0 & 0 & \dots \\ a_2 & a_2 & 0 & \dots \\ a_3 & a_3 & a_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ as the *immediate offspring* of M, for M' is itself the terraced matrix that results from removing the first row and the first column from M. Note that $M' = U^*MU$ where U is the unilateral shift.

2 Main Result

Theorem 2.1. (a) *M' inherits from M the property of hyponormality.*
(b) *M is compact if and only if M' is compact.*

Proof. (a) We must show that $[(M')^*, M'] \equiv (M')^*M' - M'(M')^* \geq 0$. Critical to the proof is the fact that $(M')^*\, M' = U^*\{(M^*M)U\}$, which can be verified by computing that both sides of the equation are equal to the reverse-L-shaped matrix $\begin{pmatrix} b_1 & b_2 & b_3 & \dots \\ b_2 & b_2 & b_3 & \dots \\ b_3 & b_3 & b_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ where $b_n = \sum_{k=n}^{\infty} |a_k|^2$; also, it can be verified that

$$M'(M')^* = \begin{pmatrix} |a_1|^2 & a_1\overline{a_2} & a_1\overline{a_3} & \dots \\ \overline{a_1}a_2 & 2\,|a_2|^2 & 2a_2\overline{a_3} & \dots \\ \overline{a_1}a_3 & 2\overline{a_2}a_3 & 3\,|a_3|^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = (U^*M)\{(UU^*)(M^*U)\}$$

and that

$$U^*\{(MM^*)U\} = \begin{pmatrix} 2\,|a_1|^2 & 2a_1\overline{a_2} & 3a_1\overline{a_3} & \dots \\ 2\overline{a_1}a_2 & 3\,|a_2|^2 & 3a_2\overline{a_3} & \dots \\ 2\overline{a_1}a_3 & 3\overline{a_2}a_3 & 4\,|a_3|^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = (U^*M)\{I(M^*U)\}.$$

Consequently, we have

$$\begin{aligned}
[(M')^*,M'] &= (M')^*M' - M'(M')^* \\
&= U^*\{(M^*M)U\} - (U^*M)\{(UU^*)(M^*U)\} \\
&= U^*\{(M^*M)U\} - U^*\{(MM^*)U\} + U^*\{(MM^*)U\} \\
&\qquad - (U^*M)\{(UU^*)(M^*U)\} \\
&= U^*\{(M^*M)U\} - U^*\{(MM^*)U\} + (U^*M)\{I(M^*U)\} \\
&\qquad - (U^*M)\{(UU^*)(M^*U)\} \\
&= U^*\{[M^*,M]U\} + (M^*U)^*\{(I - UU^*)(M^*U)\}.
\end{aligned}$$

Since M is hyponormal (by hypothesis) and $I - UU^* \geq 0$, we find that

$$\begin{aligned}
&\langle\ [(M')^*,\ M']\ f\ ,f\rangle = \\
&= \langle\ [M^*,M]\ Uf,Uf\rangle + \langle((I - UU^*)(M^*U)\ f,\ (M^*U)\ f\rangle \\
&\geq\quad 0 + 0 = 0 \qquad\qquad \text{for all } f \text{ in } \ell^2.
\end{aligned}$$

This completes the proof of part (a).

(b) We prove only one direction. Suppose M' is compact. It follows that $UM'U^*$ is also compact. Note that $M - UM'U^*$ has nonzero entries only in the first column; these entries are precisely the terms of the sequence $\{a_n\}$. Since M is bounded, we must have $\sum_{n=0}^{\infty} |a_n|^2 = \|Me_0\|^2 < \infty$, where e_0 belongs to the standard orthonormal basis for ℓ^2; consequently, $M - UM'U^*$ is a Hilbert-Schmidt operator on ℓ^2 and is therefore compact. Thus $M = UM'U^* + (M - UM'U^*)$ is compact, since it is the sum of two compact operators.

Corollary 2.1. *Assume M'' is the terraced matrix obtained by removing the first k rows and the first k columns from M, for some fixed positive integer $k > 1$.* (a) *M'' inherits from M the property of hyponormality.* (b) *M is compact if and only if M'' is compact.*

3 Other Results

We note that normality (occurring when M commutes with M^*) and quasinormality (occurring when M commutes with M^*M) are also inherited properties for terraced matrices, but those turn out to be trivialities. The proofs are left to the reader.

Theorem 3.1. (a) *If M is normal, then $a_n = 0$ for all $n \geq 1$ and $M' = 0$.* (b) *If M is quasinormal, then $a_n = 0$ for all $n \geq 1$ and $M' = 0$.*

In closing, we consider a question about the most famous terraced matrix, the Cesàro matrix C. Is C the immediate offspring of some other hyponormal terraced matrix; that is, does there exist a hyponormal terraced matrix A such that $C = A' = U^*AU$? The matrix A would have to be generated by $\{a_n\}$ with a_0 yet to be determined and $a_n = \frac{1}{n}$ for $n \geq 1$. Then $L = \lim_{n\to+\infty} (n+1)a_n = \lim_{n\to+\infty} \frac{n+1}{n} = 1$. From [**3**, Theorems 2.5 and 2.6] we conclude that the spectrum is $\sigma(A) = \{\lambda : |\lambda - 1| \leq 1\} \cup \{a_0\}$ and that A cannot be hyponormal since $\sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} > 1 = L^2$. Thus we see that nonhyponormality is not inherited by the immediate offspring of a terraced matrix.

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