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# Concerning the Euler totient [1]

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**Abstract**

In this paper we intend to establish several properties for the Euler totient denoted $\varphi$. This totient provides us with a series of arithmetic inequalities in relation with the functions $\tau$ and $\sigma$ respectively, where $\tau(m)$ is the number of natural divisors of $m$, and $\sigma(m)$ is the sum of natural divisors of $m$.

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## 1 Introduction

Let $m$ be a natural number, $m \geq 1$. We note with $\varphi(m)$ the number of positive integers less than or equal to $m$ that are coprime to $m$. Hence

$$\varphi(m) = card\{k|(k, m) = 1, k \leq m\}.$$

The function $\varphi$ so defined is the totient function. The totient is usually called the Euler totient or Euler's totient. It follows from the definition

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that $\varphi(1) = 1$ and $\varphi(p^k) = (p-1)p^{k-1}$ when $p$ is a prime number and $k \in \mathbb{N}^*$, also, $\varphi(2^k t) = 2^{k-1}\varphi(t)$, when $t$ is odd number.

Remark several properties of these functions, found in some books: [3], [5], [11], [12], [15].

In paper [7], Gauss proved the equality

$$\sum_{d/m} \varphi(d) = m. \tag{1}$$

$\varphi$ is also a multiplicative function; if $(m, n) = 1$, then

$$\varphi(mn) = \varphi(m)\varphi(n). \tag{2}$$

For $m > 1$ and $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ we have the relation

$$\varphi(m) = m(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_r}). \tag{3}$$

## 2 Properties of Euler's totient

**Proposition 1.** *The number $\varphi(m)$ is odd if $m = 1$ or $m = 2$, and even for every $m \geq 3$.*

**Proof.** For $m = 1$ or $m = 2$ we have $\varphi(1) = \varphi(2) = 1$. If $m \geq 3$ and $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, we obtain

$$\varphi(m) = p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \ldots p_r^{\alpha_r - 1}(p_1 - 1)(p_2 - 1) \ldots (p_r - 1),$$

and because at least one of the terms $p_1, p_2, \ldots p_r, p_1 - 1, p_2 - 1, \ldots, p_r - 1$ is even, then $\varphi(m)$ is even.

**Proposition 2.** $a_1 + a_2 + \ldots + a_{\varphi(m)} = \frac{m\varphi(m)}{2}$, *for every $m \geq 2$.*

**Proof.** Let $m$ be a natural number, $m \geq 2$, and $a_1 < a_2 < \ldots < a_{\varphi(m)}$, are the natural numbers less than or equal to $m$ that are coprime to $m$, so $(a_i, m) = 1$,for all $i \in \{1, 2, \ldots, \varphi(m)\}$.

As $(m-a_i, m) = 1$, we obtain $\{a_1, a_2, \dots, a_{\varphi(m)}\} = \{m-a_1, m-a_2, \dots, m-a_{\varphi(m)}\}$. Hence, because both sets have the same elements, the sum of either set is the same, so

$$a_1 + a_2 + \dots + a_{\varphi(m)} = m - a_1 + m - a_2, \dots, m - a_{\varphi(m)},$$

which is equivalent to $a_1 + a_2 + \dots + a_{\varphi(m)} = \frac{m\varphi(m)}{2}$.

**Proposition 3.** *For every $m \geq 3$, show that $a_1 + a_2 + \dots + a_{\varphi(m)} \equiv 0(\mod m)$.*

**Proof.** Because $a_1 + a_2 + \dots + a_{\varphi(m)} = \frac{m\varphi(m)}{2}$ and for every $m \geq 3$, $\varphi(m)$ is even, then $a_1 + a_2 + \dots + a_{\varphi(m)} \equiv 0(\text{ mod } m)$.

**Proposition 4.** *$a_i = m - a_{\varphi(m)-i+1}$, for all $i \in \{1, 2, \dots, \varphi(m)\}$*

**Proof.** From $a_1 < a_2 < \dots < a_{\varphi(m)}$, we obtain $m - a_{\varphi(m)} < \dots < m - a_2 < m - a_1$, hence, we deduce that $a_i = m - a_{\varphi(m)-i+1}$, for all
$i \in \{1, 2, \dots, \varphi(m)\}$.
**Remark.** Because $(a_i, m) = 1$ and $a_i < m$, for all $i \in \{1, 2, \dots, \varphi(m)\}$, we deduce that $a_1 = 1$ and $a_{\varphi(m)} = m - 1$.

**Proposition 5.** *For every $m \geq 3$, show that $a_1 < a_2 < \dots < a_{\frac{\varphi(m)}{2}} \leq \frac{m}{2} \leq a_{\frac{\varphi(m)}{2}+1} < a_{\frac{\varphi(m)}{2}+2} < \dots < a_{\varphi(m)}$ (without equality in both pairs) or $m-a_1 > m - a_2 > \dots > m - a_{\frac{\varphi(m)}{2}} \geq m - a_{\frac{\varphi(m)}{2}+1} > m - a_{\frac{\varphi(m)}{2}+2} > \dots > m - a_{\varphi(m)}$ (without equality in both parts).*

**Proof.** We may suppose by absurd that $a_{\frac{\varphi(m)}{2}} \leq \frac{m}{2}$ and $a_{\frac{\varphi(m)}{2}+1} < \frac{m}{2}$, then $a_{\frac{\varphi(m)}{2}} + a_{\frac{\varphi(m)}{2}+1} < m$, but $a_{\frac{\varphi(m)}{2}} + a_{\frac{\varphi(m)}{2}+1} = m$, so we find a contradiction; hence, for every $m \geq 3$, we obtain $a_1 < a_2 < \dots < a_{\frac{\varphi(m)}{2}} \leq \frac{m}{2} \leq a_{\frac{\varphi(m)}{2}+1} < a_{\frac{\varphi(m)}{2}+2} < \dots < a_{\varphi(m)}$ (without equality in both parts ).

**Proposition 6.** *For every $m \geq 3$, we obtain*

$$a_1 a_2 \dots a_{\frac{\varphi(m)}{2}} \equiv (-1)^{\frac{\varphi(m)}{2}} a_{\frac{\varphi(m)}{2}+1} a_{\frac{\varphi(m)}{2}+2} \dots a_{\varphi(m)}(\mod m).$$

**Proof.** From Proposition 4 we have $a_i = m - a_{\varphi(m)-i+1}$, for all $i \in \{1, 2, \ldots, \varphi(m)\}$, hence $a_i \equiv -a_{\varphi(m)-i+1} (\bmod m)$, for all $i \in \{1, 2, \ldots, \frac{\varphi(m)}{2}\}$, consequently $a_1 a_2 \ldots a_{\frac{\varphi(m)}{2}} \equiv (-1)^{\frac{\varphi(m)}{2}} a_{\frac{\varphi(m)}{2}+1} a_{\frac{\varphi(m)}{2}+2} \ldots a_{\varphi(m)} (\bmod m)$.

**Remark.** If $\frac{\varphi(m)}{2}$ is even, then $a_1 a_2 \ldots a_{\frac{\varphi(m)}{2}} \equiv a_{\frac{\varphi(m)}{2}+1} a_{\frac{\varphi(m)}{2}+2} \ldots a_{\varphi(m)} (\bmod m)$, and if $\frac{\varphi(m)}{2}$ is odd, then $a_1 a_2 \ldots a_{\frac{\varphi(m)}{2}} \equiv -a_{\frac{\varphi(m)}{2}+1} a_{\frac{\varphi(m)}{2}+2} \ldots a_{\varphi(m)} (\bmod m)$.

**Proposition 7.** *For every $m \geq 3$, we have*
$a_1^2 + a_2^2 + ... + a_{\varphi(m)}^2 + \ldots + 2(a_1 a_{\varphi(m)} + a_1 a_{\varphi(m)-1} + \ldots + a_{\frac{\varphi(m)}{2}} \cdot a_{\frac{\varphi(m)}{2}+1}) = \frac{m^2 \varphi(m)}{2}$.

**Proof.**From Proposition 4 we have $a_i = m - a_{\varphi(m)-i+1}$, for all $i \in \{1, 2, \ldots, \varphi(m)\}$, so $a_{\varphi(m)-i+1} = m - a_i$, for all $i \in \{1, 2, \ldots, \varphi(m)\}$, hence $2(a_1 a_{\varphi(m)} + a_1 a_{\varphi(m)-1} + \ldots + a_{\frac{\varphi(m)}{2}} \cdot a_{\frac{\varphi(m)}{2}+1}) = \sum_{i=1}^{\varphi(m)} a_i a_{\varphi(m)-i+1} = \sum_{i=1}^{\varphi(m)} a_i (m - a_i) = m \sum_{i=1}^{\varphi(m)} a_i - \sum_{i=1}^{\varphi(m)} a_i^2$, but $a_1 + a_2 + \ldots + a_{\varphi(m)} = \frac{m\varphi(m)}{2}$, therefore $a_1^2 + a_2^2 + ... + a_{\varphi(m)}^2 + \ldots + 2(a_1 a_{\varphi(m)} + a_1 a_{\varphi(m)-1} + \ldots + a_{\frac{\varphi(m)}{2}} \cdot a_{\frac{\varphi(m)}{2}+1}) = \frac{m^2 \varphi(m)}{2}$.

**Proposition 8.** *For every $i \in \{1, 2, \ldots, \varphi(m)\}$, we have the inequality* $a_i a_{\varphi(m)-i+1} \leq \frac{m^2}{4}$.

**Proof.** It is easy to see that $(\frac{m}{2} - a_i)(\frac{m}{2} - a_{\varphi(m)-i+1}) \leq 0$, which is equivalent to $\frac{m^2}{4} - \frac{m}{2}(a_i + a_{\varphi(m)-i+1}) + a_i a_{\varphi(m)-i+1} \leq 0$, but $a_i + a_{\varphi(m)-i+1} = m$, consequently $a_i a_{\varphi(m)-i+1} \leq \frac{m^2}{4}$.

**Proposition 9.** $a_1^2 + a_2^2 + ... + a_{\varphi(m)}^2 \geq \frac{m^2 \varphi(m)}{2}$.

**Proof.** Form Proposition 7 and Proposition 8 we have clearly Proposition 9, because we have the equality $2(a_1 a_{\varphi(m)} + a_1 a_{\varphi(m)-1} + \ldots + a_{\frac{\varphi(m)}{2}} \cdot a_{\frac{\varphi(m)}{2}+1}) = \sum_{i=1}^{\varphi(m)} a_i a_{\varphi(m)-i+1}$.

**Remark.** Let $d_1, d_2 \ldots, d_k$ be all the divisors of $m(d_1 = 1, d_k = m)$.
As $(a_i, m) = 1$, the numbers $d_2, d_3 \ldots, d_k$ belong to the set $\{1, 2, \ldots, m\} \setminus \{a_1, a_2, \ldots, a_{\varphi(m)}\}$, but $d_1 = 1, d_k = m$, and $1 < d_2, d_3 \ldots, d_{k-1} \leq \frac{m}{2}$, hence $max\{a_{\frac{\varphi(m)}{2}}, d_{k-1}\} \leq \frac{m}{2}$.

**Proposition 10.** $a_1 + a_2 + \ldots + a_{\frac{\varphi(m)}{2}} + \sigma(m) \leq \frac{m^2+10m+8}{8}$.

**Proof.** We remark that $max\{a_{\frac{\varphi(m)}{2}}, d_{k-1}\} \leq \frac{m}{2}$,therefore

$$a_1 + a_2 + \ldots + a_{\frac{\varphi(m)}{2}} + d_2 + d_3 + \ldots + d_{k-1} \leq 1 + 2 + \ldots + [\frac{m}{2}].$$

so

$$a_1+a_2+\ldots+a_{\frac{\varphi(m)}{2}}+\sigma(m)-m-1 \leq \frac{1}{2}[\frac{m}{2}]([\frac{m}{2}]+1) \leq \frac{1}{2}(\frac{m}{2})(\frac{m}{2}+1) = \frac{m^2+2m}{8},$$

which infers

$$a_1 + a_2 + \ldots + a_{\frac{\varphi(m)}{2}} + \sigma(m) \leq \frac{m^2+10m+8}{8}.$$

**Proposition 11.** $m^2 + m + 2 \geq m\varphi(m) + 2\sigma(m)$

**Proof.** Because $a_1+a_2+\ldots+a_{\frac{\varphi(m)}{2}}+d_2+d_3+\ldots+d_k \leq 1+2+\ldots+m$, and form the relations $a_1+a_2+\ldots+a_{\varphi(m)} = \frac{m\varphi(m)}{2}$ and $d_2+d_3+\ldots+d_k = \sigma(m)-1$, we obtain $\frac{m\varphi(m)}{2} + \sigma(m) - 1 \leq \frac{m(m+1)}{2}$,so $m^2 + m + 2 \geq m\varphi(m) + 2\sigma(m)$

**Proposition 12.** $a_1a_2\ldots a_{\varphi(m)} \leq m^{-\frac{\tau(m)}{2}} \cdot m!$.

**Proof.** We remark the inequality

$$a_1a_2\ldots a_{\varphi(m)}d_1d_2\ldots d_k \leq 1 \cdot 2 \cdot \ldots \cdot m = m!,$$

but $d_1d_2\ldots d_k = m^{\frac{\tau(m)}{2}}$, which infers $a_1a_2\ldots a_{\varphi(m)} \leq m! \cdot m^{-\frac{\tau(m)}{2}}$.

**Proposition 13.** $a_1a_2\ldots a_{\frac{\varphi(m)}{2}} \leq m^{-\frac{\tau(m)-2}{2}} \cdot [\frac{m}{2}]!$.

**Proof.** it is easy to see that

$$a_1a_2\ldots a_{\frac{\varphi(m)}{2}}d_2d_3\ldots d_{k-1} \leq 1 \cdot 2 \cdot \ldots \cdot [\frac{m}{2}] = [\frac{m}{2}]!,$$

but $d_1d_2\ldots d_k = m^{\frac{\tau(m)}{2}}$, so $d_2d_3\ldots d_{k-1} = m^{\frac{\tau(m)-2}{2}}$, hence the inequality becomes $a_1a_2\ldots a_{\frac{\varphi(m)}{2}} \leq m^{-\frac{\tau(m)-2}{2}} \cdot [\frac{m}{2}]!$.

**Proposition 14.** $m + \tau(m) \leq \sigma(m) + 1$, *for all* $m \geq 2$.

**Proof.** From Gauss's Theorem, $\sum_{d/m} \varphi(d) = m$, and, as $\varphi(m) \leq m - 1$, for all $m \geq 2$, we get $m = \sum_{d/m} \varphi(d) = \varphi(d_1) + \varphi(d_2) + \varphi(d_3) + \ldots + \varphi(d_k) \leq 1 + d_2 - 1 + d_3 - 1 + \ldots + d_k - 1 = \sigma(m) - \tau(m) + 1$, therefore $m \leq \sigma(m) - \tau(m) + 1$.

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