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Ropelength Under Linking Operation and Enzyme Action[1]

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Abstract

There are some interesting explorations about relations of ropelengths under operations on knots or links. One of them is the linking operation. Here we give a proof about relation of ropelengths under linking operation:

$$Rop(Lin(L_1, L_2)) \geq Rop(L_1) + Rop(L_2),$$

and give the necessary and sufficient condition for equality holding. Intuitively, we can see that, when linking two links together, it requires longer ropes.

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1 Definition

Definition 1.1. *Non-trivial linking operation is a disjoint union of two links L_1 and L_2, which are embedded in $\mathbb{R}^3$, such that there are no disjoint spheres S^2 containing each link. We denote it as $Lin(L_1, L_2)$.*

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We assume that each component of a link is a $C^{1,1}$ curve in $\mathbb{R}^3$ which is homeomorphic to S^1. Thickness of a link L, denoted as $Th(L)$, is the largest radius of normal disks around L, such that the disks are disjoint. An alternative definition for thickness is by global radius of curvature, which is

$$\inf_{\substack{x\neq y\neq z\neq x\\ x,y,z\in L}} r(x,y,z),$$

where $r(x,y,z)$ is the radius of the circle passing through x, y and z. Ropelength of a link L is defined as

$$Rop(L)=\frac{arclength(L)}{Th(L)}.$$

2 Theorem

We formulate the following theorem:

Given two links L_1 and L_2 in $\mathbb{R}^3$, then

$$Rop(Lin(L_1,L_2))\geq Rop(L_1)+Rop(L_2).$$

The equality holds iff

$$\frac{1}{2}dist(L_1,L_2)\geq max(Th(L_1),Th(L_2)),$$

where $dist(L_1,L_2)$ is the Euclidean distance of L_1 and L_2.

Proof. Let l_1, l_2 and l be the arclength of link L_1, L_2, and $Lin(L_1,L_2)$ respectively. Since

$$\begin{aligned}
(1)\qquad Th(L_i) &= \inf_{\substack{x\neq y\neq z\neq x\\ x,y,z\in L_i}} r(x,y,z)\\
(2)\qquad &\geq \inf_{\substack{x\neq y\neq z\neq x\\ x,y,z\in Lin(L_1,L_2)}} r(x,y,z)\\
(3)\qquad &= Th(Lin(L_1,L_2)) \text{ for } i=1,2,
\end{aligned}$$

then we have

$$\begin{aligned}
Rop(Lin(L_1, L_2)) &= \frac{l}{Th(Lin(L_1, L_2))} \\
&= \frac{l_1 + l_2}{Th(Lin(L_1, L_2))} \\
&= \frac{l_1}{Th(Lin(L_1, L_2))} + \frac{l_2}{Th(Lin(L_1, L_2))} \\
&\geq \frac{l_1}{Th(Lin(L_1))} + \frac{l_2}{Th(Lin(L_2))} \\
&= Rop(Lin(L_1) + Rop(Lin(L_2)).
\end{aligned}$$

If

$$\frac{1}{2} dist(L_1, L_2) \geq \max(Th(L_1), Th(L_2))$$

then (2) is true for equality;

Conversely, if

$$\frac{1}{2} dist(L_1, L_2) \; < \max(Th(L_1), Th(L_2)),$$

then we have

$$\inf_{\substack{x \neq y \neq z \neq x \\ x,y,z \in L_i}} r(x, y, z) \;\; \geq \inf_{\substack{x \neq y \neq z \neq x \\ x,y,z \in Lin(L_1, L_2)}} r(x, y, z) \text{ for } i = 1 \text{ or } 2,$$

i.e.

$$Th(L_1) > Th(Lin(L_1, L_2)) \text{ or } Th(L_2) > Th(Lin(L_1, L_2)),$$

since we can choose a point from each link to attain the infimum. Thus we have

$$Rop(Lin(L_1, L_2)) > Rop(L_1) + Rop(L_2).$$

So the equality holds only if

$$\frac{1}{2} dist(L_1, L_2) \geq \max(Th(L_1), Th(L_2)).$$

Remark Dr. Jason Cantarella makes a more general statement for the ropelength of the union of two links. The proof above can be applied to that general case.

3 Application

One of the motivations for studying knots and links topologically is to study DNA types and enzymology actions of enzyme which is important in understanding the processes of DNA replication, transcription, recombination etc(c.f. [S]), and to understand the transmission, inheridity, translation, formation etc of life information. Enzyme catalyzes strands of double helix of DNA to form links or knots, thus the lengths of strands grow during the process of actions of enzyme.

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References

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