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The method of the variation of constants for Riccati equations[1]

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Abstract

The classical method for solving Riccati equations uses a change which leads to a first order linear equation. We give here a new method of the variation of constants which leads directly to a equation with separable variables. Finally an example is given.

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1 Introduction

One method for solving linear equations of the form

(1.1) $$y' = b(x)y + c(x) \quad , \quad \text{with } b, c : I \subseteq \mathbb{R} \to \mathbb{R} \text{ continuous}$$

is the Lagrange method of the variation of the constants. First, the corresponding equation with separable variables

$$y' = b(x)y$$

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has solutions of the form $y = ce^{B(x)}$, where $B \in \int b$ and $c \in \mathbb{R}$. Hence the general solution of the linear equation (1.1) is $y = c(x)e^{B(x)}$, where $c(x)$ is deduced by substituting in (1.1).

Using this idea, we will give a simpler method for solving Riccati equations of the form

$$y' = a(x)y^2 + b(x)y + c(x) \quad , \quad \text{with } a, b, c : I \subseteq \mathbb{R} \to \mathbb{R} \text{ continuous} \tag{1.2}$$

than the classical method which we remember here. If y_0 is a particular solution of the equation (1.2), then

$$u = y - y_0 \tag{1.3}$$

satisfies the Bernoulli equation

$$u' = (2a(x)y_0 + b(x))u + a(x)u^2 \quad , \quad \text{with } r = 2.$$

On the natural way, denote further

$$u = z^{-1} \tag{1.4}$$

to obtain the linear equation

$$z' = -(2a(x)y_0 + b(x))z - a(x). \tag{1.5}$$

Now by substitute z in (1.3), we derive

$$\frac{1}{z} = y - y_0 \quad , \quad \text{so } y = y_0 + \frac{1}{z}.$$

In fact, this is the classical method for solving the Riccati equation (1.2). If y_0 is a particular solution, then the substitution $y = y_0 + \frac{1}{z}$ leads to a linear equation of the first order.

2 Main Result

Next we give a method which transforms directly the Riccati equation into a equation with separable variables. As we mentioned, the solution of that linear equation (1.5) is of the form

$$z = c(x)e^{-v(x)} \quad , \quad \text{where } \ v \in \int (2a(x)y_0 + b(x))dx.$$

Now by substitute z in (1.3)-(1.4), we derive

$$u(x)e^{v(x)} = y - y_0 \quad , \quad \text{or } y = y_0 + u(x)e^{v(x)},$$

with the renotation $u(x) = \frac{1}{c(x)}$. We can state:

Theorem 2.1 *Assume that the Riccati equation (1.2) has a particular solution y_0. Then the general solution of the Riccati equation (1.2) is of the form*

$$y = y_0 + u(x)e^{v(x)},$$

where $v(x) \in \int (2a(x)y_0 + b(x))dx$ and $u(x)$ can be deduced by substituting in (1.2). More precisely, $u(x)$ is the general solution of the equation with separable variables

$$u'(x) = a(x)e^{v(x)}u^2(x). \tag{2.1}$$

Proof. Let us substitute $y = y_0 + u(x)e^{v(x)}$ in the Riccati equation (1.2)

$$y_0' + u'(x)e^{v(x)} + u(x)(2a(x)y_0 + b(x))e^{v(x)} =$$

$$= a(x)\left(y_0^2 + 2u(x)y_0e^{v(x)} + u^2(x)e^{2v(x)}\right) + b(x)\left(y_0 + u(x)e^{v(x)}\right) + c(x).$$

y_0 is particular solution, so we reduce $y_0' = a(x)y_0^2 + b(x)y_0 + c(x)$ to obtain

$$u'(x)e^{v(x)} + u(x)(2a(x)y_0 + b(x))e^{v(x)} =$$

$$= a(x)\left(2u(x)y_0e^{v(x)} + u^2(x)e^{\ 2v(x)}\right) + b(x)u(x)e^{v(x)}.$$

Further, by dividing by $e^{v(x)}$ and other reductions terms, we derive

$$u'(x) = a(x)u^2(x)e^{v(x)},$$

so we are done.

Further, we can solve the equation (2.1) with separable variables,

$$\frac{u'(x)}{u^2(x)} = a(x) \cdot e^{v(x)} \Leftrightarrow \frac{d}{dx}(-\frac{1}{u(x)}) = a(x)e^{v(x)}$$

It follows that

$$\frac{1}{u(x)} = -\int a(x) \cdot e^{v(x)} dx$$

so we can state the following result which is the direct formula of the general solution of the Riccati equation (1.2):

Theorem 2.2 *Assume that the Riccati equation (1.2) has a particular solution y_0. Then the general solution of the Riccati equation (1.2) is of the form*

(2.2)

$y = y_0 - \frac{e^{v(x)}}{w(x)}$ *where* $v(x)$ *is a primitive of the function* $2a(x)y_0 + b(x)$ *and*

$w(x)$ *is any primitive of the function* $a(x) \cdot e^{v(x)}$.

Now we can remark that the solving of a Riccati equation is reduced to a direct substitution in the formula (2.2), so we do not need calculations for each example of Riccati equations; we purely can replace in the formula (2.2) in the equivalent form

$$y = y_0 - \frac{e^{\int_{x_0}^{x} (2a(s)y_0(s) + b(s))ds}}{c + \int_{x_0}^{x} a(s)e^{\int_{x_0}^{s} (2a(t)y_0(t) + b(t))dt}} ds, \; c \in \mathbb{R} \qquad (2.3)$$

3 An example

Let us consider the equation

$$y' = \alpha y^2 + \frac{\beta}{x} \cdot y + \frac{\gamma}{x^2}, \quad x > 0 \tag{3.1}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are such that

$$(\beta + 1)^2 = 4\alpha\gamma \neq 0.$$

That type of equation has the particular solution $y_0 = \frac{m}{x}$, where $m \in \mathbb{R}$ is the unique solution of the quadratic equation

$$\alpha m^2 + (\beta + 1)m + \gamma = 0, \quad \text{so} \;\; m = -\frac{\beta + 1}{2\alpha}.$$

Note that

$$2m\alpha + \beta = -1.$$

Using the classical way with the notation $y = \frac{m}{x} + \frac{1}{z}$, the unknown function z satisfies the linear equation

$$z' = \frac{1}{x} \cdot z - \alpha.$$

After two steps, using the Lagrange's method, we obtain $z = c - \alpha \ln x$. After replace also $m = -\frac{\beta+1}{2\alpha}$, the general solution of the equation (3.1) is

$$y = -\frac{\beta + 1}{2\alpha x} + \frac{1}{c - \alpha \ln x} \quad , \quad c \in \mathbb{R} \tag{3.2}$$

Now, by using the formula (2.3) we obtain the solution directly, with $x_0 = 1$,

$$y = \frac{m}{x} - \frac{e^{\int_1^x (\frac{2m}{s} + \frac{\beta}{s}) \mathrm{d}s}}{c + \displaystyle\int_1^x \alpha e^{\int_1^s (\frac{2m}{t} + \frac{\beta}{2}) \mathrm{d}t} \mathrm{d}s} = \frac{m}{x} - \frac{e^{\int_1^x (\frac{2\alpha m}{s} + \frac{\beta}{s}) \mathrm{d}s}}{c + \displaystyle\int_1^x \alpha e^{\int_1^s (\frac{2m}{t} + \frac{\beta}{2}) \mathrm{d}t} \mathrm{d}s} =$$

$$= \frac{m}{x} - \frac{e^{\int_1^x} - \dfrac{1}{s} \mathrm{d}s}{c + \displaystyle\int_1^x \alpha e^{\int_1^s - \frac{1}{t} \mathrm{d}t} \mathrm{d}s} = \frac{m}{x} - \frac{\frac{1}{x}}{c + \alpha \ln x} = \frac{m}{x} - \frac{1}{x(c + \alpha \ln x)},$$

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