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# On fourth order simultaneously zero-finding method for multiple roots of complex polynomial equations[1]

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**Abstract**

In this paper, we present and analyse fourth order method for finding simultaneously multiple zeros of polynomial equations. S. M. Ilič and L. Rančič modified cubically convergent Ehrlich_Aberth method to fourth order for the simultaneous determination of simple zeros [5]. We generalize this method to the case of multiple zeros of complex polynomial equations. It is proved that the method has fourth order convergence. Numerical tests show its efficient computational behaviour in the case of multiple real/complex roots of polynomial equations.

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# 1 Introduction

The methods for simultaneous finding of all roots of polynomials are very popular as compared to the methods for individual finding of the roots.These methods have a wider region of convergence and are more stable, (see, [2,6-7,9-12]) and references cited therein. For fourth order simple zero-finding simultaneous methods,(see, [4,8,13,15-16]).

S. M. Ilič and L. Rančič modified cubically convergent Ehrlich-Aberth method to fourth order for simultaneous finding of simple complex zeros of polynomial equations [5]. We generalise this method to the case of multiple zeros of complex polynomial equations. It is proved that the method has fourth order convergence, if the roots have known multiplicities. Recently, X. Zhang, H. Peng and G. Hu established a fifth order zero-finding method for the simultaneous determination of simple complex zeros of polynomial equations[15]. However, in case of multiple zeros, it has linear convergence as is also obvious from the numerical tests. Results of numerical tests show efficient computational behaviour of our method in case of multiple real/complex zeros of complex polynomial.

The method and its convergence analysis is considered in Section 2, where as results of numerical tests are presented in Sectin 3. Section 4 contains conclusion.

# 2 The method and its convergence analysis

Let us consider a monic algebraic polynomial $P$ of degree $n$ having zeros $w_j$ with multiplicities $\alpha_j$, such that $\sum_{j=1}^{m} \alpha_j = n$,

$$P(z) = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0 = \prod_{j=1}^{m} (z - w_j)^{\alpha_j}. \tag{1}$$

We propose the following method for finding the multiple zeros of complex polynomial (1):

$$\tilde{z}_i = z_i - \frac{\alpha_i}{\frac{1}{N_i} - \sum_{\substack{j=1 \\ j\neq i}}^{m} \frac{\alpha_j}{z_i - z_j} + \sum_{\substack{j=1 \\ j\neq i}}^{m} \frac{\alpha_j^2 N_j}{(z_i - z_j)^2}}, \tag{2}$$

where $N_i = \frac{P(z_i)}{P'(z_i)}$ is the Newton's correction. The method (2) is the generalization of the method presented by S. M. Ilič and L. Rančič [5] to the case of multiple zeros of a complex polynomial.We name this method as $NMM$-method. We claim that the $NMM$-method is of convergence order four.

First, let us introduce the notations,

(3)

$$d = \min_{\substack{i,j \\ i\neq j}} |w_i - w_j|, q = \frac{2n-1}{d} \text{ and } \sum, \sum_{j\neq i}, \text{ instead of} \sum_{j=1}^{m}, \sum_{\substack{j=1 \\ j\neq i}}^{m}, \text{ respectively.}$$

Further, suppose that the conditions,

$$|\epsilon_i| < \frac{d}{2n-1} = \frac{1}{q}, \ (i = 1, ..., m) \tag{4}$$

hold for all $i$, where $\epsilon_i = z_i - w_i$. We assume here that $n \geq 3$. We prove the following lemma:

**Lemma 1.** *Let $z_1, ..., z_m$ be the distinct approximations to the zeros $w_1, ..., w_m$ respectively. Also, let $\tilde{\varepsilon}_i = \tilde{z}_i - w_i$,where $\tilde{z}_i$ is the new approximation produced by the NMM-method (2). If (4) holds, then the following inequalities also hold:*

(*i*) $\left|\tilde{\epsilon}_i\right| \leq \frac{q^3}{(n-1)} |\epsilon_i|^2 \sum_{j\neq i} |\epsilon_j|^2,$

(*ii*) $\left|\tilde{\epsilon}_i\right| < \frac{d}{2n-1} = \frac{1}{q}, \ (i = 1, ..., m).$

**Proof.** Considering (3), we get

$$(5) \qquad \begin{aligned} |z_i - w_j| &= |(w_i - w_j) + (z_i - w_i)| \geq |w_i - w_j| - |z_i - w_i| \\ &> d - \frac{d}{2n-1} = \frac{2n-2}{q}, \end{aligned}$$

Considering (4) and (5), we get

$$(6) \qquad \begin{aligned} |z_i - z_j| &= |(z_i - w_j) + (w_j - z_j)| \geq |z_i - w_j| - |w_j - z_j| \\ &> \frac{2n-2}{q} - \frac{1}{q} = \frac{2n-3}{q}. \end{aligned}$$

Defining the notation

$$\sum_{1,i} = \sum_{j \neq i} \frac{1}{z_i - w_j},$$

we have

$$\sum_{1,i} \alpha_j = \sum_{j \neq i} \frac{\alpha_j}{z_i - w_j}.$$

Thus, using (5), we have

$$\left| \sum_{1,i} \alpha_j \right| \leq \sum_{j \neq i} \frac{\alpha_j}{|z_i - w_j|} < \frac{q}{2(n-1)} \sum_{j \neq i} \alpha_j = \frac{q}{2(n-1)} (n - \alpha_i).$$

Since $n - \alpha_i < n - 1$ for all $i$, we have

$$(7) \qquad \left| \sum_{1,i} \alpha_j \right| < \frac{q(n-1)}{2(n-1)} = \frac{q}{2}.$$

Also

$$\begin{aligned} \frac{1}{N_i} &= \frac{P'(z_i)}{P(z_i)} = \sum_{1,i} \frac{\alpha_j}{z_i - w_j} = \frac{\alpha_i}{z_i - w_i} + \sum_{1,i} \alpha_j \\ &= \frac{\alpha_i}{\varepsilon_i} + \sum_{1,i} \alpha_j = \frac{\alpha_i + \varepsilon_i \sum_{1,i} \alpha_j}{\varepsilon_i}. \end{aligned}$$

Now, using (4) and (7) in the above result, we have:

$$|N_i| = \left|\frac{|\varepsilon_i|}{\alpha_i + \varepsilon_i \sum_{1,i} \alpha_j}\right| < \frac{|\varepsilon_i|}{1 - |\varepsilon_i| \left|\sum_{1,i} \alpha_j\right|} < \frac{\frac{1}{q}}{1 - \frac{1}{q}.\frac{q}{2}}, \tag{8}$$

i.e.,

$$|N_i| < \frac{2}{q}. \tag{9}$$

From (2), we have

$$\begin{aligned}
\tilde{\epsilon}_i &= \tilde{z}_i - w_i = z_i - \frac{\alpha_i}{\frac{1}{N_i} - \sum\limits_{j\neq 1} \frac{\alpha_j}{z_i - z_j} + \sum\limits_{j\neq 1} \frac{\alpha_j^2 N_j}{(z_i - z_j)^2}} - w_i \\
&= \epsilon_i - \frac{\alpha_i}{\frac{\alpha_i}{\epsilon_i} + \sum\limits_{j\neq 1} \frac{\alpha_j}{z_i - w_j} - \sum\limits_{j\neq 1} \frac{\alpha_j}{z_i - z_j} + \sum\limits_{j\neq 1} \frac{\alpha_j^2 N_j}{(z_i - z_j)^2}} \\
&= \epsilon_i - \frac{\alpha_i \epsilon_i}{\alpha_i + \epsilon_i \sum\limits_{j\neq 1} \left[\frac{-\alpha_j (z_j - w_j)(z_i - z_j) + \alpha_j^2 N_j (z_i - w_j)}{(z_i - w_j)(z_i - z_j)^2}\right]},
\end{aligned}$$

Using Newton's correction, we have

$$\begin{aligned}
\tilde{\epsilon}_i &= \epsilon_i - \frac{\alpha_i \epsilon_i}{\alpha_i + \epsilon_i \sum\limits_{j\neq 1} \alpha_j \left[\frac{-\epsilon_j (z_i - z_j) + \left(\frac{\alpha_i \alpha_j}{\alpha_j + \epsilon \sum_{1,j} \alpha_i}\right)(z_i - w_j)}{(z_i - w_j)(z_i - z_j)^2}\right]} \\
&= \epsilon_i - \frac{\alpha_i \epsilon_i}{\alpha_i + \epsilon_i \sum\limits_{j\neq 1} \alpha_j \left[\frac{\epsilon_j \left\{-(z_i - z_j)\left(\alpha_j + \epsilon_j \sum_{1,j} \alpha_i\right) + \alpha_j z_i - \alpha_j w_j\right\}}{(z_i - w_j)(z_i - z_j)^2 \left(\alpha_j + \epsilon_j \sum_{1,j} \alpha_i\right)}\right]} \\
&= \epsilon_i - \frac{\epsilon_i}{1 + \frac{\epsilon_i}{\alpha_i} \sum\limits_{j\neq 1} \alpha_j \epsilon_j \left[\frac{\alpha_j (z_j - w_j) - (z_i - z_j)\epsilon_j \sum_{1,j} \alpha_i}{(z_i - w_j)(z_i - z_j)^2 \left(\alpha_j + \epsilon_j \sum_{1,j} \alpha_i\right)}\right]} \\
&= \epsilon_i - \frac{\epsilon_i}{1 + \frac{\epsilon_i}{\alpha_i} \sum\limits_{j\neq 1} \alpha_j \epsilon_j \left[\frac{\alpha_j \epsilon_j - (z_j - z_j)\epsilon_j \sum_{1,j} \alpha_i}{(z_i - w_j)(z_i - z_j)^2 \left(\alpha_j + \epsilon_j \sum_{1,j} \alpha_i\right)}\right]} \\
&= \epsilon_i - \frac{\epsilon_i}{1 + \frac{\epsilon_i}{\alpha_i} \sum\limits_{j\neq 1} \alpha_j \epsilon_j^2 A_{ij}},
\end{aligned}$$

where

$$A_{i\ j}=\frac{\alpha_j-(z_i-z_j)\sum_{1,j}\alpha_i}{(z_i-w_j)(z_i-z_j)^2\left(\alpha_j+\epsilon_j\sum_{1,j}\alpha_i\right)}$$

implies

$$\tilde{\epsilon}_i=\frac{\epsilon_i+\frac{\epsilon_i^2}{\alpha_i}\sum\limits_{j\neq 1}\alpha_j\epsilon_j^2A_{ij}-\epsilon_i}{1+\frac{\epsilon_i^2}{\alpha_i}\sum\limits_{j\neq 1}\alpha_j\epsilon_j^2A_{ij}},$$

or

$$\tilde{\epsilon}_i=\frac{\frac{\epsilon_i^2}{\alpha_i}\sum\limits_{j\neq 1}\alpha_j\epsilon_j^2A_{ij}}{1+\frac{\epsilon_i^2}{\alpha_i}\sum\limits_{j\neq 1}\alpha_j\epsilon_j^2A_{ij}}. \tag{10}$$

From (8), we have

$$|N_j|=\frac{|\epsilon_j|}{\left|\alpha_j+\epsilon_j\sum_{i,j}\alpha_i\right|}<\frac{2}{q},$$

implies

$$\frac{1}{\left|\alpha_j+\epsilon_j\sum_{i,j}\alpha_i\right|}<\frac{2}{q\,|\epsilon_j|}$$

Now using equations (5), (6) and (7), and the above result, we have

$$\begin{aligned}|A_{i\ j}| &\leq \frac{|\alpha_j|+|z_i-z_j|\left|\sum_{i,j}\alpha_i\right|}{|z_i-w_j|\,|z_i-z_j|^2\left|\alpha_j+\epsilon_j\sum_{1,\ j}\alpha_i\right|}\\ &= \frac{\frac{|\alpha_j|}{|z_i-z_j|}+\left|\sum\limits_{1,j}\alpha_i\right|}{|z_i-w_j|\,|z_i-z_j|\left|\alpha_j+\epsilon_j\sum_{1,j}\alpha_i\right|}\\ &\leq \frac{n\frac{q}{2n-3}+\frac{q}{2}}{\left(\frac{2n-2}{q}\right)\left(\frac{2n-3}{q}\right)\frac{q}{2}\,|\epsilon_j|}=\frac{qq^2}{q}\frac{2n+(2n-3)}{2(n-1)\,(2n-3)^2\,|\epsilon_j|}\\ &= \frac{q^2}{2(n-1)}\frac{1}{|\epsilon_j|}\frac{4n-3}{(2n-3)^2}.\end{aligned}$$

Since $\dfrac{4n-3}{(2n-3)}, n \geq 3$ is monotonically decreasing sequence, so that finding the least upper bound for $n \geq 3$ of the sequence, we have

$$|A_{i\ j}| < \frac{q^2}{2(n-1)} \frac{1}{|\epsilon_j|} \tag{11}$$

and

$$\begin{aligned} \left|\frac{\epsilon_i}{\alpha_i}\sum_{j\neq 1}\alpha_j\epsilon_j^2 A_{i\ j}\right| &\leq \left|\frac{\epsilon_i}{\alpha_i}\right|\sum_{j\neq i}\alpha_j.\left|\epsilon_j\right|^2\left|A_{i\ j}\right| < \frac{|\epsilon_i|}{\alpha_i}\sum_{j\neq i}\alpha_j.\left|\epsilon_j\right|^2\frac{q^2}{2(n-1)}.\frac{1}{|\epsilon_j|} \\ &= \frac{|\epsilon_i|}{\alpha_i}\sum_{j\neq i}\alpha_j\left|\epsilon_j\right|.\frac{q^2}{2(n-1)} < \frac{1}{q}.\sum_{j\neq i}\alpha_j.\frac{1}{q}.\frac{q^2}{2(n-1)} \\ &< \frac{1}{q}.\frac{q}{2(n-1)}\sum_{j\neq i}\alpha_j < \frac{1}{2(n-1)}(n-\alpha_i) \quad , \end{aligned}$$

since $\alpha_i \geq 1 \Longrightarrow \frac{1}{\alpha_i} \leq 1$ and $\dfrac{|\epsilon_i|}{\alpha_i} \leq \dfrac{1}{q}$ for all $i$. Using $\sum\limits_{j\neq i}\alpha_j = n-\alpha_i < n-1$ for all $i$, we obtain

$$\left|\frac{\epsilon_i}{\alpha_i}\sum_{j\neq 1}\alpha_j\epsilon_j^2 A_{i\ j}\right| \leq \frac{1}{2(n-1)}(n-1) = \frac{1}{2}. \tag{12}$$

Also further, using (12), imples

$$\left|1+\frac{\epsilon_i}{\alpha_i}\sum_{j\neq i}\alpha_j\epsilon_j^2 A_{i\ j}\right| \geq 1 - \left|\frac{\epsilon_i}{\alpha_i}\sum_{j\neq i}\alpha_j\epsilon_j^2 A_{i\ j}\right| > \frac{1}{2} \tag{13}$$

Finally, using equation (11), (13) in (10), we get

$$\left|\widetilde{\epsilon_i}\right| < \frac{|\epsilon_i|^2\sum\limits_{j\neq i}\alpha_j\left|\epsilon_j\right|^2\dfrac{q^2}{2(n-1)}\dfrac{1}{|\epsilon_j|}}{\dfrac{1}{2}} = \frac{q^2}{(n-1)}\left|\epsilon_i\right|^2\sum_{j\neq i}\alpha_j\left|\epsilon_j\right|. \tag{14}$$

This completes, the proof of Lemma 1(i). Now from (14), we obtain

$$\begin{aligned} \left|\widetilde{\epsilon_i}\right| &< \frac{q^2}{(n-1)}\frac{1}{q^2}\sum_{j\neq i}\alpha_j\frac{1}{q} = \frac{1}{(n-1)q}\sum_{j\neq i}\alpha_j \\ &= \frac{1}{(n-1)q}(n-\alpha_i). \end{aligned}$$

Since $\sum_{j\neq i}\alpha_j = n - \alpha_i < n - 1$ for all $i$, we have $\left|\widetilde{\epsilon}_i\right| < \frac{1}{(n-1)q}(n-1) = \frac{1}{q}$, namely

$$|\epsilon_i| < \frac{d}{2n-1} = \frac{1}{q} \implies \left|\widetilde{\epsilon}_i\right| < \frac{1}{q}$$

and hence Lemma 1 (ii) is also valid.

Let $z_1^{(0)}, ..., z_m^{(0)}$ be reasonably good initial approximations to the zeros $w_1, ..., w_m$ of the polynomial $P$, and let $\epsilon_i^{(k)} = z_i^{(k)} - w_i$, where $z_1^{(k)}, ..., z_m^{(k)}$be the approximtions obtained in the $k$th iterative step by the simultaneous method (NMM-method).

Using Lemma 1, we now state the main convergence theorem concerned with $NMM$-method.

**Theorem 1.** *Under the conditions*

$$\left|\epsilon_i^{(0)}\right| = \left|z_i^{(0)} - w_i\right| < \frac{d}{2n-1} = \frac{1}{q},\ (i = 1, ..., m), \tag{15}$$

*the NMM-method is convergent with the convergence order four.*

**Proof.** In Lemma 1 (i) we established result (14) under the conditions (4). Using the same argument under condition (15) of Theorem 1, we have from (14)

$$\left|\epsilon_i^{(1)}\right| \leq \frac{q^3}{(n-1)}\left|\epsilon_i^{(0)}\right|^2 \sum_{j\neq 1}\alpha_j \left|\epsilon_i^{(0)}\right|^2 < \frac{1}{q},\ (i = 1, ..., m).$$

So by Lemma 1 (ii), we have:

$$\left|\epsilon_i^{(0)}\right| < \frac{d}{2n-1} = \frac{1}{q} \Rightarrow \left|\epsilon_i^{(1)}\right| < \frac{d}{2n-1} = \frac{1}{q},\ (i = 1, ..., m).$$

Using the mathematical induction, we can prove that the condition (15) implies

$$\left|\epsilon_i^{(k+1)}\right| \leq \frac{q^3}{(n-1)}\left|\epsilon_i^{(k)}\right|^2 \sum_{j\neq 1}\alpha_j \left|\epsilon_j^{(k)}\right|^2 < \frac{1}{q}, \tag{16}$$

for each $k = 0, 1, ...$ and $i = 1, ..., m$.

Putting $\left|\epsilon_i^{(k)}\right| = \dfrac{t_i^{(k)}}{q}$, (16) becomes

$$\left|t_i^{(k+1)}\right| \leq \frac{\left(t_i^{(k)}\right)^2}{(n-1)} \sum_{j\neq 1} \alpha_j \left(t_j^{(k)}\right)^2, \ (i = 1, ..., m). \tag{17}$$

Let $t^{(k)} = \max\limits_{1\leq i\leq m} t_i^{(k)}$. Then from condition (15), it follows that $q\left|\epsilon_i^{(0)}\right| = t_i^{(0)} \leq t^{(0)} < 1$ for $i = 1, ..., m$, and from (17), we have $t_i^{(k)} < 1$ for each $k = 0, 1, ...$ and $i = 1, ..., m$. Thus, from (17), we get

$$\begin{aligned} \left|t_i^{(k+1)}\right| &\leq \frac{\left(t_i^{(k)}\right)^2}{(n-1)} (n - \alpha_j) \left(t^{(k)}\right)^2 \\ &< \left(t_j^{(k)}\right)^2 \frac{(n-1)}{(n-1)} \left(t^{(k)}\right)^2 \leq \left(t^{(k)}\right)^4. \end{aligned} \tag{18}$$

This shows that the sequences $\left\{t_i^{(k)}; i = 1, ..., m\right\}$ converge to 0. Consequently, the sequences $\left\{\left|\epsilon_i^{(k)}\right|\right\}$ also converge to 0, i.e., $z_i^{(k)} \to w_i$ for all $i$ as $k$ increases. Finally, from (18) it can be concluded that the method (2) (NMM-method) has convergence order four.

## 3 Numerical Tests

We consider here some numerical examples of algebraic polynomials with repeated real and complex zeros to demonstrate the performance of fourth order method (2) (NMM-method).

We use the abbreviations as GHN(10), GHN(11) and GHN(12) to refer to the formulae of convergence order two, three and three for multiple zeros in [8] and ZPH to refer to the formula of convergence order five for distinct zeros in [15].

All the computations are performed using Mapple 7.0. We take $\epsilon = 10^{-18}$ as tolerance and use the following stopping criteria for estimating the zeros:

$$\left| x_i^{n+1} - x_i^n \right| < \epsilon.$$

Firstly, taking the multiplicities equal to one in our method (2), we re-estimated the examples from [5]. We got the same estimates as by the fourth order convergent method in [5] for distinct zeros.

Secondly, numerical tests for the algebraic polynomials with real and complex repeated zeros from [8] are provided in tables 3.1(a) to 3.1(b). The roots obtained by the methods GHN(10), GHN(11) and GHN(12) are accurate to 18 digits in tables 3.1(a) to 3.1(b), where as the roots obtained by the NMM-method are accurate to 20 to 30 digits in table 3.1(a) and accurate to 21 to 32 digits in table 3.1(b).

Thirdly, the NMM-method is also compared with ZPH-method[15]. We got the roots accurate to 64 digits at the first iteration, where as the ZPH-method obtained roots accurate to 2 digits at fifth iteration.

| | | | | |
|---|---|---|---|---|
| **Table 3.1(a)** : | | | | |
| **Example 1** : | | | | |
| $z^{13} + 10z^{12} - 90z^{11} - 1000z^{10} + 3425z^9 + 39174z^8 + 81200z^7 - 741920z^6$ | | | | |
| $+1425120z^5 + 6500160z^4 - 15697152z^3 - 15966720z^2 + 66873600z$ | | | | |
| $= (z-5)^3 (z-2)^4 (z+3)(z+6)^5$ | | | | |
| **Exact Root**: $\alpha = (5, 2, -3, -6)$ | | | | |
| **Initial Point** | **Number of iterations for different methods** | | | |
| $x_0$ | $GHN(10)*$ | $GHN(11)*$ | $GHN(12)*$ | $NMM**$ |
| $(5.9, 2.7, -3.9, -6.7)$ | 7 | 5 | 5 | 3 |
| $(5.6, 1.5, -2.7, -6.5)$ | 6 | 4 | 4 | 3 |
| $(5.5, 1.4, -2.6, -6.3)$ | 6 | 4 | 4 | 3 |
| $(5.4, 2.2, -2.8, -5.9)$ | 6 | 3 | 4 | 3 |
| $*$Absolute Errors are equal to$10^{-18}$ | | | | |
| $**$Absolute Errors lie between$10^{-18} to 10-30$ | | | | |

| **Table 3.1(b)** : | | | | |
|---|---|---|---|---|
| **Example 2** : | | | | |
| $z^7-3z^6+5z^5-7z^4+7z^3-5z^2+3z-1=(z-i)^2\,(z+i)^2\,(z-1)^3=\left(z^2+1\right)^2(z-1)^3$ | | | | |
| **Exact Root**: $\alpha=(i,-i,1)$ | | | | |
| **Initial Point** | **Number of iterations for different methods** | | | |
| $x_0$ | $GHN(10)*$ | $GHN(11)*$ | $GHN(12)*$ | $NMM**$ |
| $(0.1-0.8i,0.1+0.8i,0.8-0.2i)$ | 5 | 4 | 4 | 3 |
| $(0.2-0.8i,0.2+0.8i,0.7-0.2i)$ | 23 | 5 | 12 | 3 |
| $(0.3-0.8i,0.2+0.8i,0.9-0.3i)$ | 11 | 4 | 6 | 3 |
| $(0.1-0.9i,0.3+0.85i,0.8-0.2i)$ | 6 | 3 | 4 | 3 |
| $*$Absolute Errors are equal to$10^{-18}$ | | | | |
| $**$Absolute Errors lie between$10^{-31}to10^{-32}$ | | | | |

| **Table 3.2** | | | | |
|---|---|---|---|---|
| **Example 3** : | | | | |
| $z^4-4z^3+6z^2-4z+1=(z-1)^4$ | | | | |
| **Absolute error by NMM**=$0.000000E+00$.Accuracy upto 64 digits in first iteration | | | | |
| **Absolute error by ZPH** | | | | |
| **Iteration** | $e_1^{(k)}$ | $e_2^{(k)}$ | $e_3^{(k)}$ | $e_4^{(k)}$ |
| $k=1$ | $0.220767E+00$ | $0.205550E+00$ | $0.205545E+00$ | $0.203993E+00$ |
| $k=2$ | $0.104195E+00$ | $0.100922E+00$ | $0.996648E-01$ | $0.101043E+00$ |
| $k=3$ | $0.495872E-01$ | $0.490962E-01$ | $0.484233E-01$ | $0.492193E-01$ |
| $k=4$ | $0.237278E-01$ | $0.237661E-01$ | $0.235053E-01$ | $0.237900E-01$ |
| $k=5$ | $0.548168E-02$ | $0.553296E-02$ | $0.550699E-02$ | $0.551863E-02$ |

# 4 Conclusion

From the numerical comparison, we observe that the NMM-method is compareable with fourth order methods in case of distinct zeros. It has got better performance in case of multiple zeros over the third order methods for multiple zeros and higher order methods for distinct zeros.

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