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A best approximation property of the generalized spline functions

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Abstract

In the introduction of this paper is presented the definition of the generalized spline functions as solutions of a variational problem and are shown some theorems regarding to the existence, uniqueness and characterization. The main result of this article consist in a best approximation property satisfied by the generalized spline functions in the context of the spaces, operator and interpolatory set involved.

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1 Preliminaries

Definition 1 *Let E_1 be a real linear space, $(E_2, \|.\|_2)$ a normed real linear space, $T : E_1 \to E_2$ an operator and $U \subseteq E_1$ a non-empty set. The problem* *of finding the elements $s \in U$ which satisfy*

$$\|T(s)\|_2 = \inf_{u \in U} \|T(u)\|_2, \tag{1}$$

is called the general spline interpolation problem, corresponding to the set U.

A solution of this problem, provided that exists, is named general spline interpolation element, corresponding to the set U.

The set U is called interpolatory set.

In the sequel we assume that E_1 is a real linear space, $(E_2, (.\,,.)_2, \|.\|_2)$ is a real Hilbert space, $T : E_1 \to E_2$ is a linear operator and $U \subseteq E_1$ is a non-empty convex set.

Lemma 1 *$T(U) \subseteq E_2$ is a non-empty convex set.*

The proof follows directly from the linearity of the operator T, taking into account that U is a non-empty set.

Theorem 1 (Existence Theorem) *If $T(U) \subseteq E_2$ is a closed set, then the general spline interpolation problem (1) (corresponding to U) has at least a solution.*

The proof is shown in the papers [1, 3].

For every element $s \in U$ we define the set

$$U(s) := U - s. \tag{2}$$

Lemma 2 *For every element $s \in U$ the set $U(s)$ is non-empty ($0_{E_1} \in U(s)$).*

The result follows directly from the relation (2).

Theorem 2 (Uniqueness Theorem) *If $T(U) \subseteq E_2$ is closed set and exists an element $s \in U$ solution of the general spline interpolation problem (1) (corresponding to U), such that $U(s)$ is linear subspace of E_1, then the following statements are true*

i) *For any elements $s_1, s_2 \in U$ solutions of the general spline interpolation problem (1) (corresponding to U) we have*

$$s_1 - s_2 \in Ker(T) \cap U(s); \tag{3}$$

ii) *The element $s \in U$ is the unique solution of the general spline interpolation problem (1) (corresponding to U) if and only if*

$$Ker(T) \cap U(s) = \{0_{E_1}\}. \tag{4}$$

A proof is presented in the papers [1, 2].

Theorem 3 (Characterization Theorem) *An element $s \in U$ is solution of the general spline interpolation problem (1) (corresponding to U) if and only if $T(s)$ is the unique element in $T(U)$ of the best approximation for 0_{E_2}.*

For a proof see the paper [1].

Lemma 3 *For every element $s \in U$ the set $T(U(s))$ is non-empty ($0_{E_2} \in T(U(s))$).*

This result is a consequence of Lemma 2.

Lemma 4 *If an element $s \in U$ has the property that $U(s)$ is linear subspace of E_1, then $T(U(s))$ is linear subspace of E_2.*

The property follows directly from the linearity of the operator T.

Theorem 4 (Characterization Theorem) *An element $s \in U$, such that $U(s)$ is linear subspace of E_1, is solution of the general spline interpolation problem (1) (corresponding to U) if and only if*

$$(T(s), T(\widetilde{u}))_2 = 0, \quad (\forall)\ \widetilde{u} \in U(s). \tag{5}$$

A proof is shown in the papers [1, 3].

For every element $s \in U$ we consider the set

$$\mathcal{S}(s) := \{v \in E_1 \mid (T(v), T(\widetilde{u}))_2 = 0,\ (\forall)\ \widetilde{u} \in U(s)\}. \tag{6}$$

Proposition 1 *For every element $s \in U$ the set $\mathcal{S}(s)$ has the following properties*

i) *$\mathcal{S}(s)$ is non-empty set ($0_{E_1} \in \mathcal{S}(s)$);*

ii) *$\mathcal{S}(s)$ is linear subspace of E_1;*

iii) *$Ker(T) \subseteq \mathcal{S}(s)$;*

iv) *$U(s) \cap \mathcal{S}(s) \subseteq Ker(T)$;*

v) *$Ker(T) \cap U(s) \subseteq \mathcal{S}(s)$;*

vi) *$U(s) \cap \mathcal{S}(s) = Ker(T) \cap U(s)$.*

For a proof see the paper [1].

Theorem 5 (Characterization Theorem) *An element* $s \in U$*, such that* $U(s)$ *is linear subspace of* E_1*, is solution of the general spline interpolation problem (1) (corresponding to* U*) if and only if*

$$s \in \mathcal{S}(s). \tag{7}$$

The result is a consequence of Theorem 4.

2 Main result

Lemma 5 *For every element* $s \in U$ *the set* $(T(U(s)))^{\perp}$ *has the following properties*

i) $(T(U(s)))^{\perp}$ *is non-empty set* $(0_{E_2} \in (T(U(s)))^{\perp})$*;*

ii) $(T(U(s)))^{\perp}$ *is linear subspace of* E_2*;*

iii) $(T(U(s)))^{\perp}$ *is closed set;*

iv) $(T(U(s))) \cap (T(U(s)))^{\perp} = \{0_{E_2}\}$.

This result follows directly from the property of the orthogonality, taking into account Lemma 3.

Lemma 6 *An element* $s \in U$*, such that* $U(s)$ *is linear subspace of* E_1*, is solution of the general spline interpolation problem (1) (corresponding to* U*) if and only if*

$$T(s) \in (T(U(s)))^{\perp}. \tag{8}$$

Proof. From Theorem 4 it follows that an element $s \in U$, such that $U(s)$ is linear subspace of E_1, is solution of the general spline interpolation problem (1) (corresponding to U) if and only if

$$(9)\qquad (T(s), T(\widetilde{u}))_2 = 0, \quad (\forall)\ \widetilde{u} \in U(s).$$

On the other hand we have

$$(10)\qquad \{(T(s), T(\widetilde{u}))_2 \mid \widetilde{u} \in U(s)\} = \{(T(s), \widetilde{t})_2 \mid \widetilde{t} \in T(U(s))\}.$$

Taking into account the equality (10), we deduce that the formula (9) is equivalent with

$$(11)\qquad (T(s), \widetilde{t})_2 = 0, \quad (\forall)\ \widetilde{t} \in T(U(s)),$$

i.e.

$$(12)\qquad T(s) \in (T(U(s)))^{\perp}.$$

Consequently, an element $s \in U$, such that $U(s)$ is linear subspace of E_1, is solution of the general spline interpolation problem (1) (corresponding to U) if and only if

$$(13)\qquad T(s) \in (T(U(s)))^{\perp}.$$

□

Lemma 7 *For every element $s \in U$ the following equality holds*

$$(14)\qquad T(U) - T(s) = T(U(s)).$$

The proof is based on the linearity of the operator T.

Theorem 6 *If an element $s \in U$, such that $U(s)$ is linear subspace of E_1, is solution of the general spline interpolation problem (1) (corresponding to U), then the following inequality is true*

$$(15) \quad \|T(u) - T(s)\|_2 \leq \|T(u) - \widetilde{w}\|_2, \quad (\forall)\ u \in U,\ (\forall)\ \widetilde{w} \in (T(U(s)))^{\perp},$$

with equality if and only if $\widetilde{w} = T(s)$.

Proof. Let $u \in U$, $\widetilde{w} \in (T(U(s)))^{\perp}$ be arbitrary elements.

Using the properties of the inner product $(.\,,.)_2$, we deduce

$$(16) \qquad \|T(u) - \widetilde{w}\|_2^2 = \|(T(u) - T(s)) + (T(s) - \widetilde{w})\|_2^2 =$$

$$= \|T(u) - T(s)\|_2^2 + 2(T(u) - T(s), T(s) - \widetilde{w})_2 + \|T(s) - \widetilde{w}\|_2^2.$$

As $u \in U$ it obtains

$$(17) \qquad T(u) \in T(U),$$

therefore

$$(18) \qquad T(u) - T(s) \in T(U) - T(s).$$

Taking into account that $s \in U$, from Lemma 7 it follows that

$$(19) \qquad T(U) - T(s) = T(U(s)).$$

Using the formula (19), the relation (18) becomes

$$(20) \qquad T(u) - T(s) \in T(U(s)).$$

Because $s \in U$, from Lemma 5 ii) it obtains that $(T(U(s)))^{\perp}$ is linear subspace of E_2. On the other hand, as $s \in U$, such that $U(s)$ is linear subspace of E_1, is solution of the general spline interpolation problem (1) (corresponding to U), using Lemma 6 we deduce $T(s) \in (T(U(s)))^{\perp}$. Also, we have $\widetilde{w} \in (T(U(s)))^{\perp}$. Consequently, it follows that

$$T(s) - \widetilde{w} \in (T(U(s)))^{\perp}. \tag{21}$$

From relations (20), (21) we deduce

$$(T(u) - T(s), T(s) - \widetilde{w})_2 = 0. \tag{22}$$

Substituting the formula (22) in the equality (16), it follows that

$$\|T(u) - \widetilde{w}\|_2^2 = \|T(u) - T(s)\|_2^2 + \|T(s) - \widetilde{w}\|_2^2. \tag{23}$$

The relation (23) implies

$$\|T(u) - T(s)\|_2 \leq \|T(u) - \widetilde{w}\|_2, \tag{24}$$

with equality if and only if $\|T(s) - \widetilde{w}\|_2 = 0$, i.e. $\widetilde{w} = T(s)$. □

Theorem 7 *If an element $s \in U$, such that $U(s)$ is linear subspace of E_1, is solution of the general spline interpolation problem (1) (corresponding to U), then*

$$\|T(u) - T(s)\|_2 = \inf_{\widetilde{w} \in (T(U(s)))^{\perp}} \|T(u) - \widetilde{w}\|_2, \quad (\forall)\ u \in U, \tag{25}$$

i.e. $T(s)$ is the unique element in $(T(U(s)))^{\perp}$ of the best approximation for $T(u)$, $(\forall)\ u \in U$.

This result follows directly from Theorem 6.

References

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