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Remarks on Voronovskaya's theorem

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Abstract

The present note discusses various quantitative forms of Vorvonovskaya's 1932 result dealing with the asymptotic behavior of the classical Bernstein operators. In particular the relationship between a result of Sikkema and van der Meer and an alternative approach of the authors ist discussed.

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In a recent paper [4] the well-known theorem of Voronovskaya for the classical Bernstein operators B_n was stated in the following form.

Theorem 1 *For $f \in C^2[0,1], x \in [0,1]$ and $n \in \mathbb{N}$ one has*

$$\left| n \cdot [B_n(f;x) - f(x)] - \frac{x(1-x)}{2} \cdot f''(x) \right| \leq \frac{x(1-x)}{2} \cdot \tilde{\omega}\left(f''; \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \right).$$

Here $\tilde{\omega}$ is the least concave majorant of ω, the first order modulus of continuity, satisfying

$$\omega(f;\epsilon) \leq \tilde{\omega}(f;\epsilon) \leq 2\omega(f;\epsilon), \epsilon \geq 0.$$

The above inequality follows from a more general asymptotic statement which was inspired by results of Bernstein [2] and Mamedov [6]. This is given in

Theorem 2 *Let $q \in \mathbb{N}_0$, $f \in C^q[0,1]$ and $L : C[0,1] \to C[0,1]$ be a positive linear operator. Then*

$$\left| L(f;x) - \sum_{r=0}^{q} L((e_1 - x)^r; x) \cdot \frac{f^{(r)}(x)}{r!} \right|$$
$$\leq \frac{L(|e_1 - x|^q; x)}{q!} \tilde{\omega}\left(f^{(q)}; \frac{L(|e_1 - x|^{q+1}; x)}{(q+1)L(|e_1 - x|^q; x)} \right).$$

The following remarks are obvious:

Remark 1 *Both asymptotic statements (supposing $L = L_n$, $n \in \mathbb{N}$, in Theorem 2) are in quantitative from due to the appearence of $\tilde{\omega}$.*

Remark 2 *In Theorem 1 the (absolute) moments $L((e_1 - x)^r; x)$ and $L(|e_1 - x|^r; x)$ are computed and/or manipulated in order to arrive at more instructive quantities. Of course this is not possible in Theorem 2 unless one makes additional assumptions on L.*

Remark 3 *In Theorem 1 the limit $\frac{x(1-x)}{2} \cdot f''(x)$ is explicitely given. The inequality of Theorem 2 requires extra considerations to arrive at a comparable statement.*

Remark 4 *Thinking of Theorem 2 as an asymptotic expansion (supposing again that* $L = L_n$*,* $n \in \mathbb{N}$*), this expansion is "complete" in the sense that* $q \in \mathbb{N}_0$ *is arbitrary.*

In contrast to that, the expansion of Theorem 1 is "non-complete".

Remark 5 *Both inequalities above do not give information about the asymptotic behaviour of quantities such as*

$$n[(B_n f)^{(k)}(x) - f^{(k)}(x)] \text{ for } k \geq 1.$$

That this is also a meaningful problem was shown in recent papers by Floater [3] and Abel and Heilmann [1], Theorem 3.3, for example.

A very interesting complete asymptotic expansion (in quantitative form) was already given some 30 years ago by Sikkema and van der Meer [8].

Theorem 3 *Let* $WC^q[0,1]$ *denote the set of all functions on* $[0,1]$ *whose q-th derivative is piecewise continuous,* $q \geq 0$*. Moreover, let* (L_n) *be a sequence of positive linear operators* $L_n : WC^q[0,1] \to C[0,1]$ *satisfying* $L_n(e_0; x) = 1$*. Then for all* $f \in fC^q[0,1], q \in \mathbb{N}_0, x \in [0,1]$*,* $n \in \mathbb{N}$ *and* $\delta > 0$ *one has*

$$\left| L_n(f;x) - f(x) - \sum_{r=1}^{q} \frac{L_n((e_1 - x)^r; x)}{r!} \cdot f^{(r)}(x) \right| \leq c_{n,q}(x,\delta) \cdot \omega(f^{(q)};\delta).$$

Here $c_{n,q}(x,\delta) = \delta^q \cdot L_n\left(s_{q,\mu}\left(\frac{e_1 - x}{\delta}\right); x\right)$*,*

$$\mu = \frac{1}{2} \text{ if } L_n((e_1 - x)^q; x) \geq 0,$$

$$\mu = -\frac{1}{2} \text{ if } L_n((e_1 - x)^q; x) < 0,$$

$$s_{q,\mu}(u) = \frac{1}{q!}\left(\frac{1}{2}\cdot |u|^q + \mu u^q\right) + \frac{1}{(q+1)!}\{b_{q+1}(|u|) - b_{q+1}(|u| - [|u|])\}.$$

b_{q+1} is the Bernoulli polynomial of degree $q+1$ and $[t] = \max\{z \in Z : z \le t\}$.

Moreover, the functions $c_{n,q}(x,\delta)$ are best possible for each $f \in C^q[0,1]$, $x \in [0,1], n \in \mathbb{N}$ and $\delta > 0$.

In the sequel we will deal with the case $q = 2$ only and furthermore assume that $L_n(e_1; x) = x$. The above theorem then implies the inequality given in

Corollary 1

$$\left| L_n(f;x) - f(x) - \frac{1}{2}\cdot L_n((e_1 - x)^2; x)\cdot f''(x)\right| \le c_{n,2}(x,\delta)\cdot \omega(f'',\delta),$$

where

$$\begin{aligned} c_{n,2}(x;\delta) &= \delta^2 \cdot L_n\left(s_{2,\frac{1}{2}}\left(\frac{e_1 - x}{\delta}\right); x\right) \\ s_{2,\frac{1}{2}}(u) &= \frac{1}{2}u^2 + \frac{1}{6}\{b_3(|u|) - b_3(|u| - [|u|])\}, \\ b_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x. \end{aligned}$$

As an alternative inequality we propose the one given in

Theorem 4 *Let $L : C[0,1] \to C[0,1]$ be a positive linear operator satisfying $Le_i = e_i, i = 0,1$. Then for any $f \in C^2[0,1], x \in [0,1]$ and $\delta > 0$ we have*

$$\begin{aligned} &\left| L(f;x) - f(x) - \frac{1}{2}\cdot L((e_1 - x)^2; x)\cdot f''(x)\right| \\ &\le \frac{1}{2}\cdot \max\left\{ L((e_1 - x)^2; x), \frac{1}{3\delta}L(|e_1 - x|^3; x)\right\}\cdot \tilde{\omega}(f'';\delta) \\ &\le \max\left\{ L((e_1 - x)^2; x), \frac{1}{3\delta}\cdot L(|e_1 - x|^3; x)\right\}\cdot \omega(f'',\delta). \end{aligned}$$

Proof Proceeding as in the considerations preceding Theorem 6.2 in [5] it can be seen that for $f \in C^2[0,1]$ fixed and $g \in C^3[0,1]$ arbitrary one gets

$$\left| L(f;x) - f(x) - \frac{1}{2} L((e_1-x)^2;x) \cdot f''(x) \right|$$

$$\leq L((e_1-x)^2;x) \cdot \left\{ ||(f-g)''|| + \frac{1}{6} \cdot \frac{L(|e_1-x|^3;x)}{L((e_1-x)^2;x)} \cdot \frac{2}{\delta} \cdot \frac{\delta}{2} \cdot ||g'''|| \right\}$$

$$\leq L((e_1-x)^2;x) \cdot \max\left\{ 1; \frac{1}{3\delta} \cdot \frac{L(|e_1-x|^3;x)}{L((e_1-x)^2;x)} \right\} \cdot \left\{ ||(f-g)''|| + \frac{\delta}{2} ||g'''|| \right\}.$$

Passing to the infimum over $g \in C^3[0,1]$ then implies

$$\left| L(f;x) - f(x) - \frac{1}{2} L((e_1-x)^2;x) \cdot f''(x) \right|$$

$$\leq \max\left\{ L((e_1-x)^2;x); \frac{1}{3\delta} \cdot L(|e_1-x|^3;x) \right\} \cdot K\left(\frac{\delta}{2}, f''; C[0,1], C^1[0,1] \right)$$

$$= \frac{1}{2} \max\left\{ |L((e_1-x)^2;x); \frac{1}{3\delta} L(|e_1-x|^3;x) \right\} \cdot \tilde{\omega}(f'';\delta).$$

Here we used the fact that for $f \in C[0,1]$ and $\delta > 0$ one has

$$K\left(\frac{\delta}{2}, f; C[0,1], C^1[0,1] \right) := \inf\left\{ ||f-g|| + \frac{\delta}{2} \cdot ||g'|| : g \in C^1[0,1] \right\} = \frac{1}{2} \tilde{\omega}(f;\delta).$$

See [7] for a proof of this. The second inequality of Theorem 4 is a consequence of $\tilde{\omega}(f;\delta) \leq 2 \cdot \omega(f;\delta)$. □

In order to compare the quality of our estimate with that of Sikkema and van der Meer we consider the classical Bernstein operators as an example.

Example 1 *For the Bernstein operators B_n there holds*

$$c_{n,2}(x,\delta) = \delta^2 \cdot B_n\left(s_{2,\frac{1}{2}}\left(\frac{e_1-x}{\delta} \right); x \right) \leq \frac{1}{2} \cdot \frac{x(1-x)}{n} \left\{ 1 + \frac{1}{\delta} \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \right\}.$$

Proof. First recall that

$$s_{2,\frac{1}{2}}(u) = \frac{1}{2}u^2 + \frac{1}{6} \cdot \{b_3(|u|) - b_3(|u| - [|u|])\}.$$

We put $t = |u| \geq 0$ and claim that

$$b_3(t) - b_3(t - [t]) = 3t^2[t] - 3t[t]^2 + [t]^3 - 3t[t] + \frac{3}{2}[t]^2 + \frac{1}{2}[t] \leq t^2[t].$$

Clearly this is true of $0 \leq t < 1$. So let $t \geq 1$.

We divide the two sides of the inequality by $[t] \geq 1$ and multiply by 2. Then the above inequality is equivalent to

$$6t^2 - 6t[t] + 2[t]^2 - 6t + 3[t] + 1 \leq 2t^2,$$

or

$$4t^2 - 6t + 1 \leq 6t[t] - 2[t]^2 - 3[t].$$

Now choose $k \in \mathbb{N}$ such that $k \leq t < k + 1$, then $[t] = k$, and the above reads

$$4t^2 - 6t + 1 \leq 6kt - 2k^2 - 3k.$$

It remains to check if this is true for all $t \in [k, k + 1)$.

For $t = k$ we get

$$4k^2 - 6k + 1 \leq 6k^2 - 2k^2 - 3k,$$

which is equivalent to $1 \leq 3k$ (fulfilled).

For $t = k + 1$ we have to show that

$$4(k + 1)^2 - 6(k + 1) + 1 \leq 6k(k + 1) - 2k^2 - 3k,$$

being equivalent to $-1 \leq k$ (fulfilled).

Hence the parabola $4t^2-6t+1$ lies below the straight line $6kt-2k^2-3k$ for $t\in[k,k+1]$ which is what we claimed above.

This implies that

$$\begin{aligned} s_{2,\frac{1}{2}}(u) &\leq \frac{1}{2}u^2+\frac{1}{6}u^2[|u|] \\ &\leq \frac{1}{2}u^2+\frac{1}{6}|u|^3. \end{aligned}$$

Hence

$$\begin{aligned} c_{n,2}(x,\delta) &\leq \delta^2\cdot B_n\left(\frac{1}{2}\cdot\frac{(e_1-x)^2}{\delta^2}+\frac{1}{6\delta^3}\cdot|e_1-x|^3;x\right) \\ &= \frac{1}{2}\left\{\frac{x(1-x)}{n}+\frac{1}{3\delta}\cdot B_n(|e_1-x|^3;x)\right\} \end{aligned}$$

Using the inequality (see [4])

$$B_n(|e_1-x|^3;x)\leq 3\cdot\sqrt{\frac{1}{n^2}+\frac{x(1-x)}{n}}\cdot B_n((e_1-x)^2;x)$$

we obtain

$$c_{n,2}(x,\delta)\leq\frac{1}{2}\cdot\frac{x(1-x)}{n}\left\{1+\frac{1}{\delta}\cdot\sqrt{\frac{1}{n^2}+\frac{x(1-x)}{n}}\right\}.$$

□

Example 2. *Choose $\delta = \sqrt{\frac{2}{n}}$. Then the theorem of Sikkema and van der Meer implies*

$$\left|B_n(f;x) - f(x) - \frac{x(1-x)}{2n}f''(x)\right|$$

$$\leq \frac{x(1-x)}{2n}\left\{1 + \sqrt{\frac{n}{2}} \cdot \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}}\right\} \cdot \omega\left(f''; \sqrt{\frac{2}{n}}\right)$$

$$\leq \left\{1 + \frac{1}{\sqrt{2}} \cdot \sqrt{1 + \frac{1}{4}}\right\} \cdot \frac{1}{2} \cdot \frac{x(1-x)}{n} \cdot \omega\left(f''; \sqrt{\frac{2}{n}}\right)$$

$$\leq 0.9 \cdot \frac{x(1-x)}{n}\omega\left(f''; \sqrt{\frac{2}{n}}\right).$$

This is better than the corresponding result of Videnskiĭ [9] published in 1985 and only for the Bernstein operators. In Videnskiĭ's book instead of 0.9 the constant is one.

We now apply Theorem 4 and arrive at

Corollary 2

$$\left|B_n(f;x) - f(x) - \frac{x(1-x)}{2n} \cdot f''(x)\right|$$

$$\leq \frac{x(1-x)}{2n} \cdot \max\left\{1, \frac{1}{\delta}\sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}}\right\} \cdot \tilde{\omega}(f''; \delta)$$

$$\leq \frac{x(1-x)}{2n} \cdot \max\left\{2, \frac{2}{\delta}\sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}}\right\} \cdot \omega(f''; \delta).$$

If the modulus $\omega(f''; \cdot)$ is concave, then the first inequality is better than what can be derived from Sikkema's and van der Meer's result because

$$\max\left\{1, \frac{1}{\delta}\sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}}\right\} \leq 1 + \frac{1}{\delta}\sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}}.$$

However, in the general case

$$\max\left\{2, \frac{2}{\delta}\sqrt{\frac{1}{n^2}+\frac{x(1-x)}{n}}\right\} \geq 1+\frac{1}{\delta}\sqrt{\frac{1}{n^2}+\frac{x(1-x)}{n}},$$

and equality is attained if and only if

$$\delta = \sqrt{\frac{1}{n^2}+\frac{x(1-x)}{n}}.$$

If we put $\hat{c}_{n,2}(x,\delta) := 1+\frac{1}{\delta}\sqrt{\frac{1}{n^2}+\frac{x(1-x)}{n}}$ and

$$d_{n,2}(x,\delta) := \max\left\{1, \frac{1}{\delta}\sqrt{\frac{1}{n^2}+\frac{x(1-x)}{n}}\right\},$$

then a possible outcome of this discussion is the following

Theorem 5 *For the Bernstein operators $B_n, n \in \mathbb{N}, f \in C[0,1], x \in [0,1]$ and $\delta > 0$ there holds*

$$\left|B_n(f;x) - f(x) - \frac{x(1-x)}{2n}f''(x)\right|$$
$$\leq \frac{x(1-x)}{2n} \cdot \min\left\{\hat{c}_{n,2}(x,\delta)\cdot\omega(f'',\delta); d_{n,2}(x,\delta)\cdot\tilde{\omega}(f'',\delta)\right\}.$$

All previous quantitative Voronovskaya theorems for the Bernstein operators and $f \in C^2[0,1]$ can be derived from Theorem 5. □

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