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A simple solution to Basel problem

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Abstract

In the following use present a simple proof of Euler's formula

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}. \tag{1}$$

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1 Introduction

The Basel problem is a famous problem in number theory, first posed by Pietro Mengoli in 1644, and solved by Leonhard Euler in 1735. The Basel problem asks for the precise sum of the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$.

2 The Proof

We present bellow a version of James D. Harper's [4] simple proof. We use the Fubini theorem for integrals and McLaurin's series expansion for $\tanh^{-1}$:

$$\frac{1}{2}\log\frac{1+y}{1-y} = \sum_{n=0}^{\infty}\frac{y^{2n+1}}{2n+1}, \qquad |y|<1.$$

We start with the equality

$$\int_{-1}^{1}\int_{-1}^{1}\frac{1}{1+2xy+y^2}\,\mathrm{d}y\mathrm{d}x = \int_{-1}^{1}\int_{-1}^{1}\frac{1}{1+2xy+y^2}\,\mathrm{d}x\mathrm{d}y \tag{2}$$

The left hand side of (2) gives:

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{1}\frac{1}{1+2xy+y^2}\,\mathrm{d}y\,\mathrm{d}x &= \int_{-1}^{1}\left.\frac{\arctan\frac{x+y}{\sqrt{1-x^2}}}{\sqrt{1-x^2}}\right|_{y=-1}^{y=1}\mathrm{d}x\\ &= \int_{-1}^{1}\frac{\pi}{2\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\pi^2}{2}.\end{aligned}$$

The right hand side of (2) yields:

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{1}\frac{1}{1+2xy+y^2}\,\mathrm{d}y\,\mathrm{d}x &= \int_{-1}^{1}\left.\frac{\log(1+2\,x\,y+y^2)}{2\,y}\right|_{x=-1}^{x=1}dy\\ &= \int_{-1}^{1}\frac{\log\frac{1+y}{1-y}}{y}dy = 2\int_{-1}^{1}\sum_{n=0}^{\infty}\frac{y^{2n}}{2n+1}dy\\ &= 4\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2},\end{aligned}$$

hence,

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8},$$

which is equivalent to (1).

For readers who would enjoy seeing more proofs, see the references.

References

[1] Dan Kalman, *Six ways to sum a series*, The College Mathematics Journal, **24**(5), 402–421.

[2] Robin Chapman, *Evaluating* $\zeta(2)$, Preprint, http: //www.maths.ex.ac.uk/∼rjc/etc/zeta2.pdf.

[3] Josef Hofbauer, *A simple proof of* $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$ *and related identities*, The American Mathematical Monthly **109**(2) (Feb., 2002) 196–200.

[4] James D. Harper, *A simple proof of* $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$, The American Mathematical Monthly **109**(6) (Jun. - Jul., 2003) 540–541.

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