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A note on a subclass of analytic functions defined by a generalized Sălăgean and Ruscheweyh operator

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Abstract

By means of Sălăgean differential operator and Ruscheweyh derivative we define a new class $\mathcal{BL}(m, \mu, \alpha, \lambda)$ involving functions $f \in \mathcal{A}_n$. Parallel results, for some related classes including the class of starlike and convex functions respectively, are also obtained.

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1 Introduction and definitions

Let $\mathcal{A}_n$ denote the class of functions of the form

(1) $$f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$$

which are analytic in the open unit disc $U=\{z:|z|<1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U, $n\in\mathbb{N}=\{1,2,...\}$.

Let $\mathcal{S}$ denote the subclass of functions that are univalent in U.

By $\mathcal{S}^{*}(\alpha)$ we denote a subclass of $\mathcal{A}_n$ consisting of starlike univalent functions of order α, $0\leq\alpha<1$ which satisfies

$$Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha,\quad z\in U. \tag{2}$$

Further, a function f belonging to $\mathcal{S}$ is said to be convex of order α in U, if and only if

$$Re\left(\frac{zf''(z)}{f'(z)}+1\right)>\alpha,\quad z\in U, \tag{3}$$

for some α, $(0\leq\alpha<1)$. We denote by $\mathcal{K}(\alpha)$, the class of functions in $\mathcal{S}$ which are convex of order α in U and denote by $\mathcal{R}(\alpha)$ the class of functions in $\mathcal{A}_n$ which satisfy

$$Re\,f'(z)>\alpha,\quad z\in U. \tag{4}$$

It is well known that $\mathcal{K}(\alpha)\subset\mathcal{S}^{*}(\alpha)\subset\mathcal{S}$.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f\prec g$, if there is a function w analytic in U, with $w(0)=0$, $|w(z)|<1$, for all $z\in U$ such that $f(z)=g(w(z))$ for all $z\in U$. If g is univalent, then $f\prec g$ if and only if $f(0)=g(0)$ and $f(U)\subseteq g(U)$.

Let S^m be the Sălăgean differential operator [8], $S^m:\mathcal{A}_n\to\mathcal{A}_n$, $n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, defined as

$$\begin{aligned} S^0f(z) &= f(z) \\ S^1f(z) &= Sf(z)=zf'(z) \\ S^mf(z) &= S(S^{m-1}f(z))=z(S^mf(z))',\quad z\in U. \end{aligned}$$

In [7] Ruscheweyh has defined the operator $R^m : \mathcal{A}_n \to \mathcal{A}_n$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= zf'(z) \\ (m+1)R^{m+1}f(z) &= z\left[R^m f(z)\right]' + mR^m f(z), \quad z \in U. \end{aligned}$$

Let D_λ^m be a generalized Sălăgean and Ruscheweyh operator introduced by A. Alb Lupaş in [1], $D_\lambda^m : \mathcal{A}_n \to \mathcal{A}_n$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, defined as

$$D_\lambda^m f(z) = (1-\lambda)R^m f(z) + \lambda S^m f(z), \quad z \in U, \ \lambda \geq 0.$$

We note that if $f \in \mathcal{A}_n$, then

$$D_\lambda^m f(z) = z + \sum_{j=n+1}^{\infty} \left(\lambda j^m + (1-\lambda)\, C_{m+j-1}^m\right) a_j z^j, \quad z \in U, \ \lambda \geq 0.$$

For $\lambda = 1$, we get the Sălăgean operator [8] and for $\lambda = 0$ we get the Ruscheweyh operator [7].

To prove our main theorem we shall need the following lemma.

Lemma 1 [*6*] *Let p be analytic in U with $p(0) = 1$ and suppose that*

$$Re\left(1 + \frac{zp'(z)}{p(z)}\right) > \frac{3\alpha - 1}{2\alpha}, \quad z \in U. \tag{5}$$

Then $Re\, p(z) > \alpha$ for $z \in U$ and $1/2 \leq \alpha < 1$.

2 Main results

Definition 1 *We say that a function $f \in \mathcal{A}_n$ is in the class $\mathcal{BL}(m, \mu, \alpha, \lambda)$, $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\mu \geq 0$, $\lambda \geq 0$, $\alpha \in [0, 1)$ if*

$$\left| \frac{D_\lambda^{m+1} f(z)}{z} \left(\frac{z}{D_\lambda^m f(z)} \right)^\mu - 1 \right| < 1 - \alpha, \qquad z \in U. \tag{6}$$

Remark 1 *The family $\mathcal{BL}(m,\mu,\alpha,\lambda)$ is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, $\mathcal{BL}(0,1,\alpha,1)\equiv \mathcal{S}^*(\alpha)$, $\mathcal{BL}(1,1,\alpha,1)\equiv\mathcal{K}(\alpha)$ and $\mathcal{BL}(0,0,\alpha,1)\equiv\mathcal{R}(\alpha)$. Another interesting subclasses are the special case $\mathcal{BL}(0,2,\alpha,1)\equiv\mathcal{B}(\alpha)$ which has been introduced by Frasin and Darus [5] , the class $\mathcal{BL}(0,\mu,\alpha,1)\equiv\mathcal{B}(\mu,\alpha)$ introduced by Frasin and Jahangiri [6], the class $\mathcal{BL}(m,\mu,\alpha,1)=\mathcal{BS}(m,\mu,\alpha)$ introduced and studied by A.Cătaş and A. Alb Lupaş [4] and the class $\mathcal{BL}(m,\mu,\alpha,0)=\mathcal{BR}(m,\mu,\alpha)$ introduced and studied by A.Cătaş and A. Alb Lupaş [1].*

In this note we provide a sufficient condition for functions to be in the class $\mathcal{BL}(m,\mu,\alpha,\lambda)$. Consequently, as a special case, we show that convex functions of order 1/2 are also members of the above defined family.

Theorem 1 *For the function $f\in\mathcal{A}_n$, $n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, $\mu\geq 0$, $\lambda\geq 0$, $1/2\leq\alpha<1$ if*

(7)
$$\frac{(m+2)(1-\lambda)R^{m+2}f(z)-(m+1)(1-\lambda)R^{m+1}f(z)+\lambda S^{m+2}f(z)}{(1-\lambda)R^{m+1}f(z)+\lambda S^{m+1}f(z)}-$$
$$\mu\frac{(m+1)(1-\lambda)R^{m+1}f(z)-m(1-\lambda)R^{m}f(z)+\lambda S^{m+1}f(z)}{(1-\lambda)R^{m}f(z)+\lambda S^{m}f(z)}+\mu\prec 1+\beta z,\quad z\in U,$$

where

$$\beta=\frac{3\alpha-1}{2\alpha},$$

then $f\in\mathcal{BL}(m,\mu,\alpha,\lambda)$.

Proof. If we consider

(8)
$$p(z)=\frac{D_\lambda^{m+1}f(z)}{z}\left(\frac{z}{D_\lambda^m f(z)}\right)^\mu,$$

then $p(z)$ is analytic in U with $p(0) = 1$. A simple differentiation yields

(9)
$$\frac{zp'(z)}{p(z)} = \frac{(m+2)(1-\lambda)R^{m+2}f(z) - (m+1)(1-\lambda)R^{m+1}f(z) + \lambda S^{m+2}f(z)}{(1-\lambda)R^{m+1}f(z) + \lambda S^{m+1}f(z)} -$$

$$\mu\frac{(m+1)(1-\lambda)R^{m+1}f(z) - m(1-\lambda)R^{m}f(z) + \lambda S^{m+1}f(z)}{(1-\lambda)R^{m}f(z) + \lambda S^{m}f(z)} - 1 + \mu.$$

Using (7) we get

$$Re\left(1 + \frac{zp'(z)}{p(z)}\right) > \frac{3\alpha - 1}{2\alpha}.$$

Thus, from Lemma 1 we deduce that

$$Re\left\{\frac{D_{\lambda}^{m+1}f(z)}{z}\left(\frac{z}{D_{\lambda}^{m}f(z)}\right)^{\mu}\right\} > \alpha.$$

Therefore, $f \in \mathcal{BL}(m, \mu, \alpha, \lambda)$, by Definition 1.

As a consequence of the above theorem we have the following interesting corollaries [2].

Corollary 1 *If* $f \in \mathcal{A}_n$ *and*

(10) $$Re\left\{\frac{9zf''(z) + \frac{7}{2}z^2f'''(z)}{f'(z) + \frac{1}{2}zf''(z)} - \frac{2zf''(z)}{f'(z)}\right\} > -\frac{5}{2}, \quad z \in U,$$

then

(11) $$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{3}{7}, \quad z \in U.$$

That is, f is convex of order $\frac{3}{7}$.

Corollary 2 *If* $f \in \mathcal{A}_n$ *and*

(12) $$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U,$$

then

$$\text{(13)} \qquad Re\, f'(z) > \frac{1}{2}, \quad z \in U.$$

In another words, if the function f is convex of order $\frac{1}{2}$ then $f \in \mathcal{BL}(0,0,\frac{1}{2},1) \equiv \mathcal{R}\left(\frac{1}{2}\right)$.

Corollary 3 *If $f \in \mathcal{A}_n$ and*

$$\text{(14)} \qquad Re\left\{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} > -\frac{3}{2}, \quad z \in U,$$

then

$$\text{(15)} \qquad Re\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2}, \quad z \in U.$$

That is, f is a starlike function of order $\frac{1}{2}$.

Corollary 4 *If $f \in \mathcal{A}_n$ and*

$$\text{(16)} \qquad Re\left\{\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)}\right\} > -\frac{1}{2}, \quad z \in U,$$

then $f \in \mathcal{BL}(1,1,1/2,1)$ hence

$$\text{(17)} \qquad Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2}, \quad z \in U.$$

That is, f is convex of order $\frac{1}{2}$.

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