

# On Positive Operators Without Invariant Sublattices

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## Abstract

By slightly modifying some of the examples given in [6], we obtain further examples of positive operators on the discrete Banach lattices  $c_0$ ,  $c$ , and  $\ell_p$  ( $1 \leq p < \infty$ ) without non-trivial closed invariant sublattices.

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## 1 Introduction

There has been an extensive research (see, for instance, [8]) over more than seventy years on the famous **invariant subspace problem**, which originally asks whether a bounded linear operator on a separable and infinite-dimensional

Hilbert space has a closed subspace, different than the trivial ones, that is mapped into itself by this operator. Albeit unsolved in its original form, namely on Hilbert spaces, there are oodles of different directions in which much has been done for this important problem of functional analysis. For general Banach spaces, due to the monumental work of P. Enflo [4], the invariant subspace problem is known to have a negative answer; moreover, it is still not known as of today whether a positive operator on an infinite-dimensional Banach lattice has a non-trivial, closed invariant subspace. Because of the extra order structure they have in addition to their being Banach spaces though, Banach lattices provide important advances when considered for the search for invariant subspaces for positive operators on them [2]. This led some researchers to distinguish, using their natural order structure, among different types of subspaces of Banach lattices. In this vein, A.K. Kitover and A.W. Wickstead have constructed in [6] examples of positive operators on the discrete Banach lattices  $c_0$ ,  $c$ , and  $\ell_p$  ( $1 \leq p < \infty$ ) without non-trivial closed invariant sublattices. It should be noted that the operators constructed by Kitover and Wickstead have non-trivial closed invariant subspaces. The purpose of this short note is to slightly modify some of the examples given in [6] and obtain further examples of positive operators without invariant sublattices. We will also show, via an example again, that our approach of taking a multiple of one of the shift operators does not work for both shifts. It should also be noted that in the proofs of the main results of the present paper, two well-known facts, namely the Kakutani-Krein Representation Theorem and that any of the spaces  $\ell_p$  ( $1 \leq p < \infty$ ),  $c_0$ , and  $c$  can be represented as a  $C(K)$ -space

for some compact Hausdorff space  $K$ , are intrinsically used. Throughout the paper the term “operator” will be synonymous with “linear operator,” and by a non-trivial subspace/sublattice is meant one that is different from  $\{0\}$  and the whole space. For all unexplained notation and terminology, we refer to [1], [3] and [7].

## 2 Definitions and heuristics

Before seeing the main results, we shall need some basic definitions from the theory of ordered vector spaces, as well as some classical representation theorems from operator theory along with their consequences.

An **ordered vector space** is a real vector space  $X$  equipped with an order relation  $\leq$  (i.e.,  $\leq$  is reflexive, anti-symmetric, and transitive) that is compatible with the algebraic structure of  $X$  in the sense that it satisfies the following two axioms:

- if  $x \leq y$ , then  $x + z \leq y + z$  for all  $z \in X$ ;
- if  $x \leq y$ , then  $\alpha x \leq \alpha y$  holds for all  $0 \leq \alpha \in \mathbb{R}$ .

A **Riesz space** or a **vector lattice** is an ordered vector space  $X$  with the additional property that for each pair of elements  $x, y$  in  $X$ , the supremum of the set  $\{x, y\}$ , denoted by  $x \vee y$ , exists in  $X$ , which in turn is equivalent to that the infimum of the set  $\{x, y\}$ , denoted by  $x \wedge y$ , exists in  $X$ . If  $x$  and  $y$  are two elements in a Riesz space  $X$  with  $x \leq y$ , then the **order interval**  $[x, y]$  is the subset defined by

$$[x, y] := \{z \in X \mid x \leq z \leq y\}.$$

A set is called **order-bounded** if it is contained in an order interval. For a Riesz space  $X$ , the subset

$$X^+ := \{x \in X \mid x \geq 0\}$$

is called the **positive cone** of  $X$  and the elements of  $X^+$  are called the **positive elements** of  $X$ . For any  $x \in X$ , we shall write

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0, \quad \text{and} \quad |x| := x \vee (-x)$$

and call the elements  $x^+, x^-$ , and  $|x|$  as the **positive part**, the **negative part**, and the **absolute value** of  $x$ , respectively. If a Riesz space  $X$  is equipped with a norm  $\|\cdot\|$  with the property that  $|x| \leq |y|$  in  $X$  implies  $\|x\| \leq \|y\|$ , then  $X$  is called a **normed Riesz space**. A norm-complete normed Riesz space, that is a normed Riesz space which is also a Banach space, is referred to as a **Banach lattice**. Note that all classical spaces of functional analysis are Banach lattices under their natural orderings. In particular, the real sequence spaces  $c_0$ ,  $c$ , and  $\ell_p$  ( $1 \leq p \leq \infty$ ) become Banach lattices under their natural coordinate-wise orderings and norms. Throughout the paper, all Banach spaces will be assumed to be real vector spaces unless otherwise stated. For more about Banach lattices, see [3] and [7].

A subspace  $E$  of a Banach lattice  $X$  is a **sublattice** if it is closed under the lattice operations: that is, for any  $x, y \in E$ , the elements  $x \vee y$  and  $x \wedge y$  belong to  $E$ . It then follows that  $x^+, x^-$  and  $|x|$  are in  $E$ . A subspace  $E$  of a Banach lattice  $X$  is an **ideal** if  $x \in E$  and  $|y| \leq |x|$  imply  $y \in E$ . It can be easily checked that every ideal is a sublattice.

An operator  $T : X \rightarrow Y$  between two ordered vector spaces is said to

be positive and denoted by  $T \geq 0$  or  $0 \leq T$ , whenever  $T(X^+) \subseteq Y^+$  holds. For an extensive treatment of positive operators, we refer the reader to [3].

A Banach lattice  $X$  is said to be an *AM-space* if  $x \wedge y = 0$  in  $X$  implies  $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ . The following classical result characterizes those Banach lattices that are *AM-spaces*; its proof can be found, for instance, in [3, Theorem 4.29].

**Theorem 1** (Kakutani–Bohnenblust & M. Krein–S. Krein). *A Banach lattice  $X$  is an AM-space with unit if and only if it is lattice isometric to some  $C(K)$  for a (unique up to homeomorphism) Hausdorff compact topological space  $K$ . In particular, a Banach lattice is an AM-space if and only if it is lattice isometric to a closed Riesz subspace of a  $C(K)$ -space.*

As a counterpart of the above result, we have the following fact.

**Theorem 2** ([5], Theorem 3). *Let  $K$  be a compact Hausdorff space,  $X$  be a closed subspace of  $C(K)$ , and*

$$\mathfrak{F} := \{(k_1, k_2, \lambda) \mid \text{for each } f \in X, f(k_1) = \lambda f(k_2) \text{ for some } k_1, k_2 \in K, \lambda \geq 0\}.$$

*Then  $X$  is a sublattice of  $C(K)$  if and only if*

$$\{f \in C(K) \mid \text{for each } (k_1, k_2, \lambda) \in \mathfrak{F}, f(k_1) = \lambda f(k_2)\} \subseteq X.$$

That the space of all convergent sequences and the space of all continuous functions on the one-point compactification of the natural numbers are isometric is a very well-known fact, which we only state here as the subject matter of the following proposition. Its proof can be found in a standard functional analysis book.

**Proposition 1** *The Banach lattices  $c$  and  $C(\mathbb{N}^*)$ , where  $\mathbb{N}^*$  is the one-point compactification of the natural numbers  $\mathbb{N}$ , are lattice isometric.*

Using Theorem 1 and Proposition 1, the closed sublattices of  $c$  can easily be characterized as follows.

**Corollary 1** *Let  $H$  be a closed vector sublattice of  $c$ . Then one of the following holds for all  $x := (x_n)_{n \in \mathbb{N}} \in H$ :*

- (1) *there exist  $\alpha \geq 0$  and  $m, n \in \mathbb{N}$  such that  $x_m = \alpha x_n$ ;*
- (2) *there exist  $\alpha \geq 0$  and  $m \in \mathbb{N}$  such that  $x_m = \alpha \lim_{n \rightarrow \infty} x_n$ ;*
- (3) *there exist  $\alpha \geq 0$  and  $m \in \mathbb{N}$  such that  $\lim_{n \rightarrow \infty} x_n = \alpha x_m$ .*

*Conversely, if some  $x \in c$  satisfies the equalities given in the constraints (1), (2) and (3) above, then  $x$  belongs to  $H$ .*

On the other hand, for the closed sublattices of  $\ell_p$  ( $1 \leq p < \infty$ ) or  $c_0$ , we have the following result. Recall that two vectors  $x$  and  $y$  in a Riesz space  $X$  are called **disjoint** if  $|x| \wedge |y| = 0$ .

**Theorem 3** ([9], 5.2). *Every closed sublattice of  $\ell_p$  with  $1 \leq p < \infty$  or  $c_0$  is the closed span of a finite or infinite sequence of disjoint positive vectors.*

Since  $c_0$  is a closed ideal in  $c$ , Theorem 2, Corollary 1 and Theorem 3 readily imply that for every  $x := (x_n)_{n \in \mathbb{N}} \in H$ , where  $H$  is a proper closed sublattice of  $\ell_p$  ( $1 \leq p < \infty$ ) or  $c_0$ , there exist  $\alpha \geq 0$  and  $m, n \in \mathbb{N}$  such that  $x_m = \alpha x_n$ .

### 3 Main results

We will give now the main results of this note, following the same lines of thought as in [6] in their proofs. It should be noted, and is straightforward to check, that all the operators constructed in the following theorems are positive. The operator given in Example 3.5 in [6] is the sum of a backward and a forward shift operator. We will show below that the conclusion therein still holds true if the backward shift operator is made into a weighted backward shift operator, with a constant weight.

**Theorem 4** *Let  $X = \ell_p$  ( $1 \leq p < \infty$ ) or  $c_0$ ,  $\alpha \geq 1$ , and  $T$  be the operator on  $X$  defined by*

$$(Tx)_n = \begin{cases} \alpha x_{n-1} + x_{n+1}, & \text{if } n > 0; \\ x_1, & \text{if } n = 0. \end{cases}$$

*Then there is no non-trivial closed  $T$ -invariant sublattice of  $X$ .*

*Proof.* By Example 3.5 in [6], one can assume that  $\alpha > 1$ . We will show that  $T$  does not have a positive eigenvector. Each element  $x \in X$  defines a function  $f(z) = \sum_{n=0}^{\infty} x_n z^n$  that is analytic on the open unit disc in the complex plane. The action of  $T$  on  $X$  corresponds to the mapping taking  $f$  to  $\alpha z f(z) + (f(z) - f(0))/z$ . If  $x$  is positive and  $Tx = \lambda x$ , then  $0 \leq \lambda \leq 1 + \alpha$  since  $\|T\| = 1 + \alpha$ . We also have

$$\alpha z f(z) + (f(z) - f(0))/z = \lambda f(z)$$

so that

$$f(z) = \frac{f(0)}{\alpha z^2 - \lambda z + 1}.$$

Thus, by scaling and taking  $f(0) = \alpha$ , one has  $f(z) = (z^2 - (\lambda/\alpha)z + 1/\alpha)^{-1}$ . For  $0 \leq \lambda \leq 1 + \alpha$ , since at least one of the roots of  $z^2 - (\lambda/\alpha)z + 1/\alpha = 0$  lies inside it,  $f$  has a pole in the open unit disc, contradicting the fact that  $f$  is analytic. This shows that  $T$  has no positive eigenvector.

Now suppose that  $H$  is a non-trivial closed  $T$ -invariant sublattice of  $X$ . First we show that there is no  $m$  such that  $x_m = 0$  for all  $x \in H$ . If  $m = 0$  then  $(Tx)_0 = x_1 = 0$  for all  $x \in H$  so we may suppose that  $m > 0$ . If there were such an  $m$ , then take  $x \in H^+$  and observe that  $(Tx)_m = \alpha x_{m-1} + x_{m+1} = 0$  so (as  $x \geq 0$ )  $x_{m-1} = x_{m+1} = 0$ . This must also hold for all  $x = x^+ - x^- \in H$ . Proceeding inductively we see that  $x_p = 0$  for all non-negative integers  $p$  so that  $H = \{0\}$ . Otherwise, if  $H \neq X$ , there are  $m, n \geq 0$  with  $m > n$  and  $\gamma > 0$  such that  $x_m = \gamma x_n$  for all  $x \in H$ . We claim that if  $x \in H$  and  $x_1, x_2, \dots, x_m$  are known then  $x$  is specified uniquely. Consider the statement  $P(p)$  that we can express  $x_k$  uniquely as a linear combination of term  $x_j$ , with  $1 \leq j \leq m$ , for all  $k \leq p$ . This is trivially true for  $p = m$ . Let us assume  $P(p)$ . Note that  $T^{p+1-m}x \in H$  as  $H$  is  $T$ -invariant and that  $(T^{p+1-m}(x))_m$  is a linear combination of  $x_k$  for  $k \leq p+1$  with the coefficient of  $x_{p+1}$  being 1. Similarly,  $(T^{p+1-m}(x))_n$  is a linear combination of  $x_k$  for  $k \leq p+n+1-m$ . As  $(T^{p+1-m}(x))_m = \gamma(T^{p+1-m}(x))_n$  we can solve for  $x_{p+1}$  in terms of  $x_k$  for  $k \leq p$  and hence express  $x_{p+1}$  as a linear combination of  $x_k$  for  $1 \leq k \leq m$ . That is, we have proved  $P(p+1)$ . It follows that  $H$  is finite-dimensional. But now Theorem 8.11 of [1] tells us that  $T$  has a positive eigenvector, which we already know to be false.  $\square$

The argument in Theorem 4 is no longer true if  $X = c$ , as  $c_0$  is a non-trivial closed ideal in that case. Interestingly, if the forward shift operator



becomes a weighted forward shift with a constant weight, the conclusion of Theorem 4 still cease to exist. The following result illustrates this fact.

**Theorem 5** *Let  $X = \ell_p$  ( $1 \leq p < \infty$ ) or  $c_0$ ,  $\alpha > 1$ , and  $T$  be the operator on  $X$  defined by*

$$(Tx)_n = \begin{cases} x_{n-1} + \alpha x_{n+1}, & \text{if } n > 0; \\ \alpha x_1, & \text{if } n = 0. \end{cases}$$

*Then  $T$  has a positive eigenvector, and consequently  $T$  has a non-trivial closed invariant sublattice.*

*Proof.* We will show that  $2\sqrt{\alpha}$  is an eigenvalue of  $T$  with the corresponding eigenvector  $x = (2n + 2)/(n + 2)$ . Indeed, for  $n = 0$ , since  $(Tx)_0 = \alpha x_1 = 2/\sqrt{\alpha}$  and  $2\sqrt{\alpha}x_0 = 2/\sqrt{\alpha}$ , we have  $(Tx)_0 = 2\sqrt{\alpha}x_0$ . Moreover, for  $n > 0$ , since

$$(Tx)_n = x_{n-1} + \alpha x_{n+1} = \frac{n}{\alpha^{\frac{n+1}{2}}} + \alpha \frac{n+2}{\alpha^{\frac{n+3}{2}}} = \frac{2n+2}{\alpha^{\frac{n+1}{2}}} = 2\sqrt{\alpha}x_n,$$

we conclude that  $(Tx)_n = 2\sqrt{\alpha}x_n$  for each  $n$ , so that  $Tx = 2\sqrt{\alpha}x$ . Thus,  $2\sqrt{\alpha}$  is an eigenvalue of  $T$  with  $x$  being the corresponding eigenvector, and therefore  $\text{span}\{x\}$  is a non-trivial closed  $T$ -invariant sublattice of  $X$ .  $\square$

The following is a weighted-shift-variant of Example 3.6 in [6].

**Theorem 6** *Let  $X = c$ ,  $\alpha \geq 1$ , and  $T$  be the operator on  $X$  defined by*

$$(Tx)_n = \begin{cases} \alpha x_{n-1} + x_{n+1} + x_0, & \text{if } n > 0; \\ \alpha x_1 + x_0, & \text{if } n = 0. \end{cases}$$

*Then there is no non-trivial closed  $T$ -invariant sublattice of  $X$ .*

*Proof.* By Example 3.6 in [6], one may assume that  $\alpha > 1$ . We will show that  $T$  does not have a positive eigenvector. As in the proof of Theorem 4, each element  $x \in X$  defines a function  $f(z) = \sum_{n=0}^{\infty} x_n z^n$  that is analytic on the open unit disc in the complex plane. Now the action of  $T$  on  $X$  corresponds to the mapping taking  $f$  to  $\alpha z f(z) + (f(z) - f(0))/z + f(0)/(1-z)$ . If  $x \in X^+$  is an eigenvector of  $T$ , then there exists a  $\lambda \geq 0$  such that  $Tx = \lambda x$ . Thus

$$\alpha z f(z) + (f(z) - f(0))/z + f(0)/(1-z) = \lambda f(z),$$

from which it follows that

$$f(z) = \frac{(1-2z)f(0)}{(1-z)(\alpha z^2 - \lambda z + 1)}.$$

If  $\lambda \geq 2\sqrt{\alpha}$  then  $\frac{\lambda - \sqrt{\lambda^2 - 4\alpha}}{2\alpha} < 1$ ; on the other hand, if  $\lambda < 2\sqrt{\alpha}$  then, as  $\lambda^2 - 4\alpha < 0$ , we have

$$\left| \frac{\lambda \mp \sqrt{\lambda^2 - 4\alpha}}{2\alpha} \right| = \frac{\sqrt{\lambda^2 + 4\alpha - \lambda^2}}{2\alpha} = \frac{1}{\sqrt{\alpha}} < 1$$

for each non-negative  $\lambda$ , therefore,  $f$  has a pole in the open unit disc, contradicting the fact that  $f$  is analytic. This shows that  $T$  has no positive eigenvector.

Now suppose that  $H$  is a non-trivial closed  $T$ -invariant sublattice of  $X$ . Arguing as in the proof of Theorem 4, we first show that there is no  $m$  such that  $x_m = 0$  for all  $x \in H$ . If  $m > 1$  and we take any  $x \in H^+$ , then  $(Tx)_m = \alpha x_{m-1} + x_{m+1} + x_0$ , so, by positivity,  $x_0 = x_{m-1} = x_{m+1} = 0$  for all  $x \in H^+$  and hence for all  $x \in H$ . Proceeding inductively we see that  $x_p = 0$  for all  $p$  (the fact that  $x_2 = 0$  will give us  $x_0 = x_1 = x_3 = 0$ ). If  $m = 1$ , then the fact that  $(Tx)_1 = (\alpha + 1)x_0 + x_2$  will give us  $x_0 = x_2 = 0$  for all

$x \in H$  in a similar way. We may now revert to the  $m > 1$  case. Finally, if  $m = 0$  then  $(Tx_0) = \alpha x_1 + x_0$  showing that  $x_1 = 0$  for all  $x \in H$  and again we may revert to a previous case. It is also impossible that  $\lim_{n \rightarrow \infty} x_n = 0$  for all  $x \in H$  as in that case we would have, for each  $x \in H^+$ ,

$$\lim_{n \rightarrow \infty} (Tx)_n = \lim_{n \rightarrow \infty} (\alpha x_{n-1} + x_{n+1} + x_0) = x_0$$

so that  $x_0 = 0$  for all  $x \in H^+$  and hence for all  $x \in H$ . But we have already seen that this is impossible.

The only possibility left, if  $H \neq X$ , is that there are  $m, n$  and  $\gamma > 0$  such that  $x_m = \gamma x_n$  for all  $x \in H$  or that there are  $m$  and  $\gamma > 0$  such that  $x_m = \gamma \lim_{n \rightarrow \infty} x_n$ . In the first of these cases, the proof that  $H$  must be finite-dimensional proceeds exactly as in the Example 3.5 of [6] and we obtain a contradiction. In the second case, notice that there can only be one such constraint as if we also have  $x_p = \beta \lim_{n \rightarrow \infty} x_n$  for some  $\beta > 0$  and for all  $x \in H$ , then  $x_m = (\alpha/\beta)x_p$  for all  $x \in H$ , which we have already established is impossible. This means that the constraint that  $x_m = \gamma \lim_{n \rightarrow \infty} x_n$  is the only possible restriction on  $H$ .

If  $m = 0$ , set  $b$  to be the sequence starting from  $\gamma, 0$  and then having all its terms 1 so that  $b \in H$ . Note that  $\lim_{n \rightarrow \infty} (Tb)_n = 3$  whilst  $(Tb)_0 = \gamma$  so that  $\gamma = 3\gamma$  which contradicts  $\gamma > 0$ . If  $m > 0$  then let  $a_m = \gamma$ ,  $a_n = 1$  for  $n \geq m + 2$  and with all other  $a_n = 0$ . This time,  $(Ta_m) = 0$  whilst  $\lim_{n \rightarrow \infty} (Tb)_n = 3$  so that  $0 \times \gamma = 3$ , which is impossible. All possible constraints, which can be possibly hold on a proper closed sublattice of  $X$ , have now been eliminated, and therefore it follows that there can be no such closed  $T$ -invariant sublattice.  $\square$

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