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Coefficient bounds for some families of starlike and convex functions of complex order

Sevtap Sümer Eker, Bilal Şeker , Mugur Acu

Abstract

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1 Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Given two functions $f, g \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

the *Hadamard product* (or *convolution*) $f * g$ is defined (as usual) by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z) \ , \ z \in \mathbb{U}.$$

For $\alpha_j \in \mathbb{C} \quad (j = 1, 2, ...q)$ and $\beta_j \in \mathbb{C} \backslash \mathbb{Z}_0^- \ ; \ \mathbb{Z}_0^- = \{0, -1, -2, ...\} \ (j = 1, 2, ...s)$ the generalized hypergeometric function ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ is defined by

$${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1)_k ... (\beta_s)_k} \frac{z^k}{k!}$$

$$\left(q \leq s + 1, \ \ q, s \in \mathbb{N}_0 = \{0, 1, 2, ...\} = \mathbb{N} \cup \{0\}\right).$$

Here, and in what follows, $(\kappa)_n$ denotes the Pochhammer symbol (or shifted factorial) defined, in terms of the Gamma function Γ, by

$$(\kappa)_n = \frac{\Gamma(\kappa + n)}{\Gamma(\kappa)} = \begin{cases} 1 & n = 0, \kappa \neq 0 \\ \kappa(\kappa + 1)...(\kappa + n - 1) & n \in \mathbb{N} \end{cases}$$

For the function

$$h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$$

the Dziok-Srivastava linear operator [2] (see also [3]) $H_s^q(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ is defined by the following Hadamard product (or convolution) :

$$H_s^q(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)f(z)=h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)*f(z)$$
$$=z+\sum_{k=2}^{\infty}\frac{(\alpha_1)_{k-1}...(\alpha_q)_{k-1}}{(\beta_1)_{k-1}...(\beta_s)_{k-1}}\frac{1}{(k-1)!}a_kz^k.$$

For notational simplicity, we write

$$H_s^q(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)f(z)=\left(H_s^q[\alpha_1]f\right)(z)$$

A function $f(z)\in\mathcal{A}$ is said to be in the class $\mathcal{S}^*(b)$ if it satisfies the following inequality:

$$Re\left[1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\right]>0 \qquad (z\in\mathbb{U},\ b\in\mathbb{C}^*=\mathbb{C}\setminus\{0\})$$

Furthermore, a function $f(z)\in\mathcal{A}$ is said to be in the class $\mathcal{C}(b)$ if it also satisfies the following inequality:

$$Re\left[1+\frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right]>0 \qquad (z\in\mathbb{U},\ b\in\mathbb{C}^*)$$

The function classes $\mathcal{S}^*(b)$ and $\mathcal{C}(b)$ were considered earlier by Nasr and Aouf [4-6] and Wiatrowski [7], respectively.

Furthermore, a function $f(z)\in\mathcal{A}$ is said to be in the class $\mathcal{SC}(b,\lambda,\gamma)$

$$Re\left[1+\frac{1}{b}\left(\frac{z[\lambda zf'(z)+(1-\lambda)f(z)]'}{\lambda zf'(z)+(1-\lambda)f(z)}-1\right)\right]>\gamma \qquad (1.2)$$

$$(f(z)\in\mathcal{A};\ 0\le\lambda\le1;\ 0\le\gamma<1;\ b\in\mathcal{C}^*;\ z\in\mathbb{U}).$$

The function class satisfying the inequality (1.2) was considered by Altıntaş et al. [1].

Let $\mathcal{SC}_{\alpha,\beta}^{q,s}(b,\lambda,\gamma)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following condition:

$$Re\left\{1+\frac{1}{b}\left(\frac{z[\lambda z(H_s^q[\alpha_1]f)'(z)+(1-\lambda)(H_s^q[\alpha_1]f)(z)]'}{\lambda z(H_s^q[\alpha_1]f)'(z)+(1-\lambda)(H_s^q[\alpha_1]f)(z)}-1\right)\right\}>\gamma, \quad (1.3)$$

$$\left(f(z)\in\mathcal{A},\ 0\leq\lambda\leq 1,\ 0\leq\gamma<1,\ b\in\mathbb{C}^*,\ q\leq s+1,\ q,s\in\mathbb{N}_0,\ z\in\mathbb{U}\right).$$

For $q=2,\ s=1,\ \alpha_1=\beta_1,\ \alpha_2=1$, we obtain the class of $\mathcal{SC}(b,\lambda,\gamma)$. Furthermore, for $q=2,\ s=1,\ \alpha_1=\beta_1,\ \alpha_2=1,\ \gamma=0$ and $\lambda=0$ the class $\mathcal{SC}_{\alpha,\beta}^{q,s}(b,\lambda,\gamma)$ is coincide the class $\mathcal{S}^*(b)$ and for $q=2,\ s=1,\ \alpha_1=\beta_1$, $\alpha_2=1,\ \gamma=0$ and $\lambda=1$ we obtain the class $\mathcal{C}(b)$.

The main object of the present investigation is to derive some coefficient bounds for functions in the subclass $\mathcal{T}_{\alpha,\beta}^{q,s}(b,\lambda,\gamma;\mu)$ of $\mathcal{A}$ which consists of functions $f(z)\in\mathcal{A}$ satisfying the following nonhomogeneous Cauchy-Euler differential equation:

$$z^2\frac{d^2w}{dz^2}+2(1+\mu)z\frac{dw}{dz}+\mu(1+\mu)w=(1+\mu)(2+\mu)g(z) \quad (1.4)$$

$$\left(w=f(z)\in\mathcal{A},\ g(z)\in\mathcal{SC}_{\alpha,\beta}^{q,s}(b,\lambda,\gamma),\ \mu\in\mathbb{R}\setminus(-\infty,-1]\right).$$

2 Main results

Theorem 1. *Let the function* $f(z) \in \mathcal{A}$ *be defined by (1.1). If the function* $f(z)$ *is in the class* $\mathcal{SC}^{q,s}_{\alpha,\beta}(b,\lambda,\gamma)$, *then*

$$|a_k| \le \frac{\prod_{j=0}^{k-2}[j+2|b|(1-\gamma)]}{[1+\lambda(k-1)]\frac{(\alpha_1)_{k-1}\dots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}}} \qquad (k \in \mathbb{N}^* = \mathbb{N}\backslash\{1\} = \{2,3,4,...\}) \tag{2.1}$$

Proof. Let the function $f(z) \in \mathcal{A}$ be given by (1.1) and let the function $F(z)$ be defined by

$$F(z) = \lambda z(H^q_s[\alpha_1]f)'(z) + (1-\lambda)(H^q_s[\alpha_1]f)(z), \qquad (f(z) \in \mathcal{A},\ 0 \le \lambda \le 1).$$

Then from (1.3) and the definition of $F(z)$ above, it is easily seen that

$$Re\left[1+\frac{1}{b}\left(\frac{zF'(z)}{F(z)}-1\right)\right] > \gamma$$

with

$$F(z) = z + \sum_{k=2}^{\infty} A_k z^k \in \mathcal{A}$$

$$\left(A_k = [1+\lambda(k-1)]\frac{(\alpha_1)_{k-1}\cdots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}}\frac{1}{(k-1)!}a_k, \qquad k \in \mathbb{N}^*\right)$$

Thus, by setting

$$\frac{1+\frac{1}{b}\left(\frac{zF'(z)}{F(z)}-1\right)-\gamma}{1-\gamma} = p(z)$$

or, equivalently,

$$zF'(z) = [1 + b(1-\gamma)(p(z)-1)]\, F(z), \tag{2.2}$$

we get

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \qquad (z \in \mathbb{U}). \tag{2.3}$$

Since

$$Re\,(p(z)) > 0, \qquad 0 \le \gamma < 1\,;\, b \in \mathbb{C}^*$$

we conclude that

$$|p_k| \le 2 \qquad (k \in \mathbb{N}).$$

We also find from (2.2) and (2.3) that

$$(k-1)A_k = b(1-\gamma)\left(p_1 A_{k-1} + p_2 A_{k-2} + \cdots + p_{k-1}\right).$$

In particular, for $k = 2, 3, 4$, we have

$$A_2 = b(1-\gamma)p_1 \ \textit{implies}\ |A_2| \le 2\,|b|\,(1-\gamma)$$

$$2A_3 = b(1-\gamma)(p_1 A_2 + p_2) \ \ \textit{implies}\ |A_3| \le \frac{2\,|b|\,(1-\gamma)[1 + 2|b|(1-\gamma)]}{2!}$$

and

$$3A_4 = b(1-\gamma)(p_1 A_3 + p_2 A_2 + p_3)$$

implies

$$|A_4| \leq \frac{2\,|b|\,(1-\gamma)[1+2|b|(1-\gamma)][2+2|b|(1-\gamma)]}{3!}$$

respectively. Using the principle of mathematical induction, we obtain,

$$|A_k| \leq \frac{\prod_{j=0}^{k-2}[j+2|b|(1-\gamma)]}{(k-1)!} \qquad (k \in \mathbb{N}^*).$$

Moreover, by the relationship between the functions $f(z)$ and $F(z)$, it is clear that

$$A_k = [1+\lambda(k-1)]\frac{(\alpha_1)_{k-1}\cdots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}}\frac{1}{(k-1)!}a_k \qquad (k \in \mathbb{N}^*).$$

and then we get

$$|a_k| \leq \frac{\prod_{j=0}^{k-2}[j+2|b|(1-\gamma)]}{[1+\lambda(k-1)]\frac{(\alpha_1)_{k-1}\cdots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}}}.$$

By choosing suitable values of the admissible parameters b, λ, γ, α and β in Theorem 1 above, we deduce the following corollaries.

Corollary 1. *(Altıntaş et al. [1]) If a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{SC}(b,\lambda,\gamma)$, then*

$$|a_k| \leq \frac{\prod_{j=0}^{k-2}[j+2|b|(1-\gamma)]}{(k-1)![1+\lambda(k-1)]} \qquad (k \in \mathbb{N}^*).$$

Corollary 2. *(Nasr and Aouf [4]) If a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}^*(b)$, then*

$$|a_k| \leq \frac{\prod_{j=0}^{k-2}[j+2|b|]}{(k-1)!} \qquad (k \in \mathbb{N}^*).$$

Corollary 3. *(Nasr and Aouf [4]) If a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{C}(b)$, then*

$$|a_k| \leq \frac{\prod_{j=0}^{k-2}[j+2|b|]}{k!} \qquad (k \in \mathbb{N}^*).$$

Theorem 2. *Let the function $f(z) \in \mathcal{A}$ be defined by (1.1). If the function $f(z)$ is in the class $\mathcal{T}^{q,s}_{\alpha,\beta}(b,\lambda,\gamma;\mu)$, then*

$$|a_k| \leq \frac{(1+\mu)(2+\mu)\prod_{j=0}^{k-2}[j+2|b|(1-\gamma)]}{[1+\lambda(k-1)]\frac{(\alpha_1)_{k-1}\cdots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}}(k+\mu)(k+1+\mu)}, \qquad (k \in \mathbb{N}^*). \tag{3.1}$$

Proof. Let $f(z) \in \mathcal{A}$ be given by (1.1). Also let

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{SC}^{q,s}_{\alpha,\beta}(b,\lambda,\gamma), \tag{3.2}$$

so

$$a_k = \frac{(1+\mu)(2+\mu)}{(k+\mu)(k+1+\mu)} b_k, \qquad (k \in \mathbb{N}^*,\ \mu \in \mathbb{R} \setminus (-\infty,-1]) \tag{3.3}$$

Thus, by using Theorem 1, we readily obtain

$$|a_k| \leq \frac{(1+\mu)(2+\mu)\prod_{j=0}^{k-2}[j+2|b|(1-\gamma)]}{[1+\lambda(k-1)]\frac{(\alpha_1)_{k-1}\cdots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}}(k+\mu)(k+1+\mu)}, \qquad (k \in \mathbb{N}^*)$$

which is precisely the assertion (3.1) of Theorem 2.

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Sevtap Sümer Eker
Dicle University
Department of Mathematics
21280 - Diyarbakır, Turkey
E-mail: sevtaps@dicle.edu.tr Bilal Şeker

Batman University
Department of Mathematics
72060 - Batman, Turkey
E-mail: bilalseker1980@gmail.com Mugur Alexandru Acu

"Lucian Blaga" University
Department of Mathematics
550012 - Sibiu, Romania
E-mails: acu_mugur@yahoo.com