General Mathematics Vol. 18, No. 1 (2010), 99–112

The degree of approximation by Bernstein operators in the knots [1]

Radu Păltănea

Abstract

We prove the inequality $\left|B_n\left(f, \frac{k}{n}\right) - f\left(\frac{k}{n}\right)\right| \leq \frac{7}{8}\omega_2\left(f, \frac{1}{\sqrt{n}}\right)$, where B_n is the Bernstein operator of order $n \geq 1$, the integer k is such that $0 \leq k \leq n$ and ω_2 denotes the usual second order modulus. Also, we give a better estimate of approximation for the point $\frac{1}{2}$.

2000 Mathematics Subject Classification: 41A10, 41A36, 41A25

Key words and phrases: Bernstein operators, second order modulus

1 Introduction. Main results

Denote by $B[0,1]$, the space of bounded real functions on the interval $[0,1]$, with the sup-norm: $\|\cdot\|$ and denote by $C[0,1]$ the subspace of continuous

[1] *Received 10 October, 2009*

Accepted for publication (in revised form) 5 December, 2009

functions. We denote the monomial functions $e_j(t) = t^j$, $j = 0, 1, 2, \ldots$ and let Π_1 the set of linear functions.

The Bernstein operators $B_n : B[0,1] \to \mathbf{R}^{[0,1]}$, $n \in \mathbf{N}$ are given by:

$$B_n(f,x) = \sum_{j=0}^{n} p_{n,j}(x) \cdot f\left(\frac{j}{n}\right), \quad f \in B[0,1],\ x \in [0,1], \tag{1}$$

$$p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}. \tag{2}$$

We express the order of approximation in terms of the second order of continuity, given by:

$$\begin{aligned} \omega_2(f,h) &= \sup\{|f(x+\rho) - 2f(x) + f(x-\rho)|,\ x \pm \rho \in [0,1], \\ &\quad 0 < \rho \le h\},\ \text{for } f \in B[0,1],\ h > 0. \end{aligned}$$

The global order of approximation is given, in the following theorem, see [4]:

Theorem A *For any $n \in \mathbf{N}$ we have*

$$\sup_{f \in B[0,1] \setminus \Pi_1} \frac{\|B_n(f) - f\|}{\omega_2\left(f, \frac{1}{\sqrt{n}}\right)} = \sup_{f \in C[0,1] \setminus \Pi_1} \frac{\|B_n(f) - f\|}{\omega_2\left(f, \frac{1}{\sqrt{n}}\right)} = 1. \tag{3}$$

Moreover, if we take into account the result given in [3], we obtain

Theorem B For any irrational number $x \in (0,1)$,

$$\sup_{n \in \mathbb{N}} \sup_{f \in C[0,1] \setminus \Pi_1} \frac{|B_n(f,x) - f(x)|}{\omega_2\left(f, \frac{1}{\sqrt{n}}\right)} = \sup_{n \in \mathbb{N}} \sup_{f \in B[0,1] \setminus \Pi_1} \frac{|B_n(f,x) - f(x)|}{\omega_2\left(f, \frac{1}{\sqrt{n}}\right)} = 1. \tag{4}$$

On the other hand, for the degree of approximation in the knots, the following result of Gonska and Zhou, [1] is known:

Theorem C *Let*

$$r = \frac{1 + \sum_{j=2}^{\infty} j^2 e^{-2(j-1)^2}}{2 + 2\sum_{j=2}^{\infty} e^{-2(j-1)^2}} \approx 0.68.$$

Then for any $\frac{1}{2} \le a < 1$, $\varepsilon > 0$, *there is an* $N(a,\varepsilon) \in \mathbf{N}$ *such that for all* $n \ge N(a,\varepsilon)$, $f \in C[0,1]$,

$$\sup_{1-a \le \frac{k}{n} \le a} \left| B_n\left(f, \frac{k}{n}\right) - f\left(\frac{k}{n}\right) \right| \le (r+\varepsilon)\cdot \omega_2\left(f, \frac{1}{\sqrt{n}}\right), \tag{5}$$

i.e., for any $\frac{1}{2} \le a < 1$ we have

$$\limsup_{n\to\infty} \sup_{1-a \le \frac{k}{n} \le a} \sup_{f \in C[0,1]\setminus \Pi_1} \frac{\left|B_n\left(f, \frac{k}{n}\right) - f\left(\frac{k}{n}\right)\right|}{\omega_2\left(f, \frac{1}{\sqrt{n}}\right)} \le r. \tag{6}$$

In connection to Theorem C, we show in this paper that there exists a constant $C < 1$ such that the inequality $\left|B_n\left(f, \frac{k}{n}\right) - f\left(\frac{k}{n}\right)\right| \le C\dot{\omega}_2\left(f, \frac{1}{\sqrt{n}}\right)$ holds uniformly for all $n \in \mathbb{N}$ and all the knots $\frac{k}{n}$, $0 \le k \le n$. More exactly, we have

Theorem 1 *The following inequality*

$$\left| B_n\left(f, \frac{k}{n}\right) - f\left(\frac{k}{n}\right) \right| \le \frac{7}{8}\omega_2\left(f, \frac{1}{\sqrt{n}}\right), \tag{7}$$

for all $n \in \mathbb{N}$, $0 \le k \le n$, $f \in B[0,1]$.

As regard to the estimate in the point $\frac{1}{2}$ we prove:

Theorem 2 *We have*

$$\limsup_{n\to\infty} \sup_{f\in B[0,1]\setminus\Pi_1} \frac{\left|B_n\left(f,\frac{1}{2}\right)-f\left(\frac{1}{2}\right)\right|}{\omega_2\left(f,\frac{1}{\sqrt{n}}\right)} \le \frac{5}{8}+\frac{1}{2\sqrt{2\pi}e^2} \le 0.652. \tag{8}$$

Note that the constant given in Theorem 2 is smaller than the constant r in Theorem C.

2 Auxiliary results

Lemma 1 *We have*

$$\frac{1}{2}\cdot p_{n,k-j}\left(\frac{k}{n}\right) \le p_{n,k+j}\left(\frac{k}{n}\right) \le p_{n,k-j}\left(\frac{k}{n}\right), \tag{9}$$

for all integers $k\ge 1$, $n\ge 2k$, $0\le j\le k$.

Proof. For the integers $k\ge 1$, $n\ge 2k$, $0\le j\le k$, denote

$$U_j^{n,k}=\frac{p_{n,k+j}\left(\frac{k}{n}\right)}{p_{n,k-j}\left(\frac{k}{n}\right)}.$$

We have

$$U_j^{n,k}=\frac{(k-j)!(n-k+j)!}{(k+j)!(n-k-j)!}\left(\frac{k}{n-k}\right)^{2j}.$$

Then for $k\ge 1$, $n\ge 2k$, $0\le j<k$ it follows

$$\begin{aligned}\frac{U_{j+1}^{n,k}}{U_j^{n,k}} &= \frac{(n-k+j+1)(k-j)}{(n-k-j)(k+j+1)}\left(\frac{k}{n-k}\right)^2\\ &= \frac{(n-k)k+(2k-n)j-j(j+1)}{(n-k)k+(n-2k)j-j(j+1)}\left(\frac{k}{n-k}\right)^2\\ &\le 1.\end{aligned}$$

Since $U_0^{n,k}=1$, for $k\ge 1$, $n\ge 2k$, we obtain

$$U_j^{n,k}\le 1,\ \text{for } k\ge 1,\ n\ge 2k,\ 0\le j\le k. \tag{10}$$

On other hand, in order to prove

$$U_j^{n,k} \geq \frac{1}{2}, \text{ for } k \geq 1,\ n \geq 2k,\ 0 \leq j \leq k, \tag{11}$$

it is sufficient to show that $U_k^{n,k} \geq \frac{1}{2}$, for $k \geq 1$, $n \geq 2k$. Put $m = n - k$. Then $k \leq m$. Denote $a_m^k = U_k^{m+k,k}$. It remains to show that

$$a_m^k \geq \frac{1}{2}, \text{ for } 1 \leq k \leq m. \tag{12}$$

We have

$$a_m^k = \frac{(k+m)!}{(2k)!(m-k)!}\left(\frac{k}{m}\right)^{2k}.$$

Consequently

$$\frac{a_{m+1}^k}{a_m^k} = \frac{k+m+1}{m-k+1}\left(\frac{m}{m+1}\right)^{2k}.$$

If we consider the function $\varphi_k(m) = \dfrac{k+m+1}{m-k+1}\left(\dfrac{m}{m+1}\right)^{2k}$, $m \in [k, \infty)$, where $k \geq 1$, we find

$$\frac{\mathrm{d}}{\mathrm{d}m}\varphi_k(m) = \frac{2k}{(m-k+1)^2} \cdot \frac{m^{2k-1}}{(m+1)^{2k+1}} \cdot (m - k^2 + 1).$$

Also,

$$\lim_{m \to \infty} \varphi_k(m) = 1.$$

Then one of the following cases is true:

i) $\varphi_k(m) \leq 1$, for $m \in [k, \infty)$;

ii) There is $m_0 \in (k, k^2 - 1)$ such that $\varphi_k(m) \geq 1$, for $m \in [k, m_0]$ and $\varphi_k(m) \leq 1$, for $m \in [m_0, \infty)$.

Consequently, we have:

$$\inf_{m\geq k} a_m^k = \min\left\{a_k^k,\ \lim_{m\to\infty} a_m^k\right\} = \min\left\{1,\ \frac{k^{2k}}{(2k)!}\right\}.$$

Finally, in order to show (12) it suffices to show that

$$\frac{(2k)!}{k^{2k}} \leq 2,\ k\in\mathbb{N},\ k\geq 1.$$

This inequality is obvious for $k=1$. Using the Stirling formula, we obtain for $k\geq 1$:

$$\frac{(2k)!}{k^{2k}} \leq 2\sqrt{k\pi}\left(\frac{2}{e}\right)^{2k} e^{\frac{1}{24}}.$$

Denoting $t_k = 2\sqrt{k\pi}\left(\frac{2}{e}\right)^{2k} e^{\frac{1}{24}}$, we obtain

$$\frac{t_{k+1}}{t_k} = \sqrt{\frac{k+1}{k}}\left(\frac{2}{e}\right)^2 \leq \sqrt{2}\left(\frac{2}{e}\right)^2 < 1.$$

So that, for $k\geq 2$,

$$\frac{(2k)!}{k^{2k}} \leq t_k < t_2 = 2\sqrt{2\pi}\left(\frac{2}{e}\right)^4 e^{\frac{1}{24}} = 1.42\ldots < 2.$$

The proof is finished.

Lemma 2 (**Sikkema [7]**) *For any $x\in[0,1]$ and any integers $1\leq s\leq n$, we have:*

$$\sum_{i=s}^{n} p_{n,i}(x)\left(\frac{i}{n}-x\right) = \binom{n-1}{s-1}x^s(1-x)^{n+1-s}. \tag{13}$$

For any $a\in\mathbb{R}$, denote by $[a]$, the integer part of a.

Lemma 3 *Let the integers k,n, such that $1\leq k\leq n/2$. Put $m=\min\{k,[\sqrt{n}]\}$. We have*

$$\sum_{j=1}^{m} p_{n,k+j}\left(\frac{k}{n}\right) + \frac{1}{2}\cdot p_{n,k}\left(\frac{k}{n}\right) > \frac{1}{4}. \tag{14}$$

Proof. First we prove the relation

$$\sum_{i=k-m}^{k+m} p_{n,i}\left(\frac{k}{n}\right) \geq \frac{3}{4}. \tag{15}$$

We consider two cases.

Case 1: $m = [\sqrt{n}]$. We have

$$\begin{aligned}\sum_{|i-k|>\sqrt{n}} p_{n,i}\left(\frac{k}{n}\right) &\leq n \sum_{|i-k|>\sqrt{n}} p_{n,i}\left(\frac{k}{n}\right)\left(\frac{i-k}{n}\right)^2 \\ &\leq n\sum_{i=0}^{n} p_{n,i}\left(\frac{k}{n}\right)\left(\frac{i-k}{n}\right)^2 = \frac{k(n-k)}{n^2} \leq \frac{1}{4}.\end{aligned}$$

Therefore relation (15) is true.

Case 2: $m = k$. Using the following identity (13) we obtain

$$\begin{aligned}\sum_{i=2k+1}^{n} p_{n,i}\left(\frac{k}{n}\right) &\leq \frac{n}{k}\sum_{i=2k+1}^{n} p_{n,i}\left(\frac{k}{n}\right)\left(\frac{i}{n}-\frac{k}{n}\right) \\ &= \binom{n-1}{2k}\left(\frac{k}{n}\right)^{2k}\left(\frac{n-k}{n}\right)^{n-2k} \\ &= \frac{(n-1)!}{(n-2k-1)!}\cdot\frac{k^{2k}}{(2k)!}\cdot\frac{1}{n^{2k}}\left(\frac{n-k}{n}\right)^{n-2k}.\end{aligned}$$

Also, we get

$$\begin{aligned}\frac{(n-1)!}{(n-2k-1)!} &= \prod_{j=1}^{k}(n-j)(n-2k-1+j) \\ &\leq \prod_{j=1}^{k}(n^2-n(2k+1)+k(k+1)) \leq (n-k)^{2k}.\end{aligned}$$

Hence we have

$$\sum_{i=2k+1}^{n} p_{n,i} \leq \frac{k^{2k}}{(2k)!}\left(\frac{n-k}{n}\right)^{n}.$$

From the inequality $(1+1/t)^{t+1} > e$, $t > 0$ and from the Stirling formula we obtain successively:

$$\sum_{i=2k+1}^{n} p_{n,i} \le \frac{k^{2k}}{(2k)!}\cdot\frac{1}{e^k} \le \frac{k^{2k}}{\sqrt{4\pi k}}\left(\frac{e}{2k}\right)^{2k}\frac{1}{e^k}$$
$$= \frac{1}{2\sqrt{\pi k}}\left(\frac{e}{4}\right)^k \le \frac{e}{8\sqrt{\pi}} \le \frac{1}{4}.$$

So, relation (15) is true in Case 2 too.

Now using Lemma 1 we obtain

$$\sum_{j=1}^{m} p_{n,k+j}\left(\frac{k}{n}\right) + \frac{1}{2}\cdot p_{n,k}\left(\frac{k}{n}\right)$$
$$\ge \frac{1}{3}\sum_{j=1}^{m}\left(p_{n,k+j}\left(\frac{k}{n}\right) + p_{n,k-j}\left(\frac{k}{n}\right)\right) + \frac{1}{3}\cdot p_{n,k}\left(\frac{k}{n}\right)$$
$$= \frac{1}{3}\sum_{i=k-m}^{k+m} p_{n,i}\left(\frac{k}{n}\right) \ge \frac{1}{4}.$$

We need also of some other results. The following lemma is a simplified version of an estimate given in [5]

Lemma 4 *Let $F : B[0,1] \to \mathbb{R}$ be a positive linear functional such that $F(e_0) = 1$ and $F(e_1) = y$, where $y \in (0,1)$. Then, for any $f \in B[0,1]$ and $0 < h \le 1/2$ we have*

$$|F(f) - f(y)| \le \left(1 + h^{-2}F((e_1 - y)^2)\right)\omega_2(f,h). \tag{16}$$

For the integers $1 \le s \le n$ and $x \in [0,1)$, denote

$$\Psi_{n,s}(x) = \sum_{j=s}^{n} p_{n,j}(x)\left(\frac{j}{n} - x\right), \tag{17}$$

and for any $x \in [0,1)$, denote $\tau(x) = \min\{i \in \mathbf{N} \mid i/n > x\}$. In [2] it is proved the following lemma.

Lemma 5 *For any number $\alpha > 0$ we have*

(18)
$$\lim_{n\to\infty} \Psi_{n,\tau\left(x+\frac{\alpha}{\sqrt{n}}\right)}(x)/\Psi_{n,\tau(x)}(x) = \exp\left(-\frac{\alpha^2}{2x(1-x)}\right),$$

uniformly with respect to x on each compact interval included in $(0,1)$.

3 Proofs of the main results

Proof of Theorem 1

Let $n \in \mathbb{N}$ and the integer $0 \le k \le n$. The inequality in (7) is obvious for $k=0$ and $k=n$. So we consider $1 \le k \le n-1$. Also using the symmetry it suffices to consider only the case $1 \le k \le n/2$. Denote $m = \min\{k, [\sqrt{n}]\}$.

Consider the positive linear functional $F_1 : B[0,1] \to \mathbb{R}$:

(19)
$$F_1(f) = \sum_{j=1}^{m} p_{n,k+j}\left(\frac{k}{n}\right)\left[f\left(\frac{k-j}{n}\right) + f\left(\frac{k+j}{n}\right)\right] + p_{n,k}\left(\frac{k}{n}\right) f\left(\frac{k}{n}\right).$$

Denote
$$T = \sum_{j=1}^{m} p_{n,k+j}\left(\frac{k}{n}\right) + \frac{1}{2}\cdot p_{n,k}\left(\frac{k}{n}\right).$$

From Lemma 3, $T \ge \frac{1}{4}$. From the definition of the second order modulus we obtain

(20)
$$\left|F_1(f) - 2Tf\left(\frac{k}{n}\right)\right| \le T\omega_2\left(f, \frac{1}{\sqrt{n}}\right), \quad f \in B[0,1].$$

Then define the linear functional $F_2 : B[0,1] \to \mathbb{R}$, $F_2(\cdot) = B_n\left(\cdot, \frac{k}{n}\right) - F_1(\cdot)$. From Lemma 1 we have $p_{n,k+j}\left(\frac{k}{n}\right) \le p_{n,k-j}\left(\frac{k}{n}\right)$, for $j \le k$. It follows that F_2 is also a positive functional. Since $F_1(e_0) = 2T$ and $B_n\left(e_0, \frac{k}{n}\right) = 1$, it follows $F_2(e_0) = 1 - 2T$. Also $F_1(e_1) = 2T\frac{k}{n}$ and $B_n\left(e_1, \frac{k}{n}\right) = \frac{k}{n}$.

Consequently $F_2(e_1) = (1-2T)\frac{k}{n}$. If we apply Lemma 4, for the functional $F = (1-2T)^{-1}F_2$ we obtain, for any $f \in B[0,1]$:

(21)
$$\left|F_2(f) - (1-2T)f\left(\frac{k}{n}\right)\right| \le \left[1 - 2T + \frac{n}{2}F_2\left(\left(e_1 - \frac{k}{n}\right)^2\right)\right]\omega_2\left(f, \frac{1}{\sqrt{n}}\right).$$

But $F_2\left(\left(e_1 - \frac{k}{n}\right)^2\right) \le B_n\left(\left(e_1 - \frac{k}{n}\right)^2, \frac{k}{n}\right) = \frac{k(n-k)}{n^3} \le \frac{1}{4n}$. Using relations (20) and (21) it follows:

$$\begin{aligned}\left|B_n\left(f,\frac{k}{n}\right) - f\left(\frac{k}{n}\right)\right| &\le \left|F_1(f) - 2Tf\left(\frac{k}{n}\right)\right| + \left|F_2(f) - (1-2T)f\left(\frac{k}{n}\right)\right| \\ &\le T\omega_2\left(f,\frac{1}{\sqrt{n}}\right) + \left(1 - 2T + \frac{1}{8}\right)\omega_2\left(f,\frac{1}{\sqrt{n}}\right) \\ &= \left(\frac{9}{8} - T\right)\omega_2\left(f,\frac{1}{\sqrt{n}}\right) \le \frac{7}{8}\omega_2\left(f,\frac{1}{\sqrt{n}}\right).\end{aligned}$$

Proof of Theorem 2

With the notation given in formula (17) we first prove

$$\lim_{n\to\infty}\sqrt{n}\Psi_{n,\tau\left(\frac{1}{2}\right)}\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{2\pi}}. \tag{22}$$

We consider two cases.

Case 1: $n = 2q$. Using formula (13) we have:

$$\begin{aligned}\sqrt{n}\Psi_{n,\tau\left(\frac{1}{2}\right)}\left(\frac{1}{2}\right) &= \sqrt{2q}\sum_{j=q+1}^{2q} p_{2q,j}\left(\frac{1}{2}\right)\left(\frac{j}{2q} - \frac{1}{2}\right) \\ &= \sqrt{2q}\binom{2q-1}{q}\left(\frac{1}{2}\right)^{2q+1} = \sqrt{2q}\binom{2q}{q}\left(\frac{1}{2}\right)^{2q+2}.\end{aligned}$$

Case 2: $n = 2q+1$. From formula (13) we have:

$$\begin{aligned}\sqrt{n}\Psi_{n,\tau\left(\frac{1}{2}\right)}\left(\frac{1}{2}\right) &= \sqrt{2q+1}\sum_{j=q+1}^{2q+1} p_{2q+1,j}\left(\frac{1}{2}\right)\left(\frac{j}{2q+1} - \frac{1}{2}\right) \\ &= \sqrt{2q+1}\binom{2q}{q}\left(\frac{1}{2}\right)^{2q+2}.\end{aligned}$$

Then

$$\lim_{n\to\infty}\sqrt{n}\Psi_{n,\tau\left(\frac{1}{2}\right)}\left(\frac{1}{2}\right)=\lim_{q\to\infty}\sqrt{2q}\binom{2q}{q}\left(\frac{1}{2}\right)^{2q+2}.$$

Using the Stirling formula we get:

$$\begin{aligned}\sqrt{2q}\binom{2q}{q}\left(\frac{1}{2}\right)^{2q+2} &= \sqrt{2q}\cdot\frac{\sqrt{4\pi q}\left(\frac{2q}{e}\right)^{2q}e^{\Theta_1(q)}}{\left[\sqrt{2\pi q}\left(\frac{q}{e}\right)^{q}e^{\Theta_2(q)}\right]^2}\left(\frac{1}{2}\right)^{2q+2}\\ &= \frac{1}{2\sqrt{2\pi}}\cdot e^{\Theta_1(q)-2\Theta_2(q)}\end{aligned}$$

where $\lim_{q\to\infty}\Theta_1(q)=0$ and $\lim_{q\to\infty}\Theta_2(q)=0$. It follows (22).

From Lemma 5 we obtain

$$\lim_{n\to\infty}\Psi_{n,\tau\left(\frac{1}{2}+\frac{1}{\sqrt{n}}\right)}\left(\frac{1}{2}\right)/\Psi_{n,\tau\left(\frac{1}{2}\right)}\left(\frac{1}{2}\right)=e^{-2}.$$

Combining with relation (22) we derive the relation:

(23) $$\lim_{n\to\infty}\sqrt{n}\Psi_{n,\tau\left(\frac{1}{2}+\frac{1}{\sqrt{n}}\right)}\left(\frac{1}{2}\right)=\frac{1}{2\sqrt{2\pi}\,e^2}.$$

Using the symmetry we can define, for $n\geq 4$

$$T_n=\sum_{\frac{n}{2}+\sqrt{n}<j\leq n}p_{n,j}\left(\frac{1}{2}\right)=\sum_{0\leq j<\frac{n}{2}-\sqrt{n}}p_{n,j}\left(\frac{1}{2}\right).$$

We have

$$\begin{aligned}T_n &\leq \sqrt{n}\sum_{\frac{n}{2}+\sqrt{n}<j\leq n}p_{n,j}\left(\frac{1}{2}\right)\left(\frac{j}{n}-\frac{1}{2}\right)\\ &= \sqrt{n}\Psi_{n,\tau\left(\frac{1}{2}+\frac{1}{\sqrt{n}}\right)}\left(\frac{1}{2}\right).\end{aligned}$$

From (23) we deduce

(24) $$\limsup_{n\to\infty}T_n\leq\frac{1}{2\sqrt{2\pi}\,e^2}.$$

For $f \in B[0,1]$, $n \geq 4$, consider the decomposition

$$B_n\left(f, \frac{1}{2}\right) = F_n^1(f) + F_n^2(f),$$

where

$$\begin{aligned} F_n^1(f) &= \sum_{\frac{n}{2}-\sqrt{n} \leq j \leq \frac{n}{2}+\sqrt{n}} p_{n,j}\left(\frac{1}{2}\right) f\left(\frac{j}{n}\right), \\ F_n^2(f) &= \sum_{0 \leq j < \frac{n}{2}-\sqrt{n}} p_{n,j}\left(\frac{1}{2}\right) f\left(\frac{j}{n}\right) + \sum_{\frac{n}{2}+\sqrt{n} < j \leq n} p_{n,j}\left(\frac{1}{2}\right) f\left(\frac{j}{n}\right). \end{aligned}$$

The linear positive functionals F_n^1 and F_n^2 satisfy the conditions $F_n^1(e_0) = 1 - 2T_n$, $F_n^1(e_1) = (1-2T_n)\frac{1}{2}$, $F_n^2(e_0) = 2T_n$ and $F_n^2(e_1) = 2T_n \cdot \frac{1}{2}$.

For $f \in B[0,1]$ we have

$$\left|B_n\left(f, \frac{1}{2}\right) - f\left(\frac{1}{2}\right)\right| \leq \left|F_n^1(f) - (1-2T_n) f\left(\frac{1}{2}\right)\right| + \left|F_n^2(f) - 2T_n f\left(\frac{1}{2}\right)\right|.$$

Using the symmetry we obtain successively:

$$\begin{aligned} & \left|F_n^1(f) - (1-2T_n) f\left(\frac{1}{2}\right)\right| \\ = & \left|\sum_{\frac{n}{2}-\sqrt{n} \leq j \leq \frac{n}{2}+\sqrt{n}} p_{n,j}\left(\frac{1}{2}\right)\left[f\left(\frac{j}{n}\right) - f\left(\frac{1}{2}\right)\right]\right| \\ \leq & \sum_{\frac{n}{2} < j \leq \frac{n}{2}+\sqrt{n}} p_{n,j}\left(\frac{1}{2}\right)\left|f\left(\frac{j}{n}\right) + f\left(\frac{n-j}{n}\right) - 2f\left(\frac{1}{2}\right)\right| \\ \leq & \sum_{\frac{n}{2} < j \leq \frac{n}{2}+\sqrt{n}} p_{n,j}\left(\frac{1}{2}\right) \cdot \omega_2\left(f, \frac{1}{\sqrt{n}}\right) \\ \leq & \frac{1}{2} \sum_{\frac{n}{2}-\sqrt{n} \leq j \leq \frac{n}{2}+\sqrt{n}} p_{n,j}\left(\frac{1}{2}\right) \omega_2\left(f, \frac{1}{\sqrt{n}}\right) \\ = & \frac{1}{2}(1-2T_n)\omega_2\left(f, \frac{1}{\sqrt{n}}\right). \end{aligned}$$

Also, Applying Lemma 4 to the functional $G = (2T_n)^{-1}F_n^2$ we obtain

$$\begin{aligned}\left|F_n^2(f) - 2T_n f\left(\frac{1}{2}\right)\right| &\leq \left(2T_n + \frac{n}{2}F_n^2\left(\left(e_1 - \frac{1}{2}\right)^2\right)\right)\omega_2\left(f, \frac{1}{\sqrt{n}}\right) \\ &\leq \left(2T_n + \frac{n}{2}B_n\left(\left(e_1 - \frac{1}{2}\right)^2, \frac{1}{2}\right)\right)\omega_2\left(f, \frac{1}{\sqrt{n}}\right) \\ &= \left(2T_n + \frac{1}{8}\right)\omega_2\left(f, \frac{1}{\sqrt{n}}\right).\end{aligned}$$

Consequently it follows:

$$\left|B_n\left(f, \frac{1}{2}\right) - f\left(\frac{1}{2}\right)\right| \leq \left(\frac{5}{8} + T_n\right)\omega_2\left(f, \frac{1}{\sqrt{n}}\right).$$

Finally, using relation (24) we obtain relation (8).

References

[1] H. Gonska and D-x Zhou, *On an extremal problem concerning Bernstein operators*, Serdica Math. J., 21, 1995, 137-150.

[2] R. Păltănea. *On an optimal constant in approximation by Bernstein operators*, Rend. Circ. Mat. Palermo, 52, Suppl., 1998, 663-686.

[3] R. Păltănea, *Estimates with second order moduli*, Rend. Circ. Mat. Palermo, 68 Suppl., 2002, 727-738.

[4] R. Păltănea, *Optimal constant in approximation by Bernstein operators*, J. Comput. Analysis Appl., 5, No. 2, 2003, 195-235.

[5] R. Păltănea, *Approximation theory using positive linear operators*, Birkhäuser, Boston, 2004.

[6] R. Păltănea, *On approximation by Bernstein operators in the knots*, Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity, 6, 2008, 91-96.

[7] P.C. Sikkema, *Der Wert einiger Konstanten in der Theorie der Approximation mit Bernstein-Polynomen*, Numerische Math., 3, 1961, 107-116.

Radu Păltănea
"Transilvania" University of Braşov
Department of Mathematics
Str. Eroilor, 29, Braşov 500 036, Romania
e-mail: radupaltanea@yahoo.com