General Mathematics Vol. 18, No. 1 (2010), 113–117

Note of the constants of Landau [1]

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Abstract

We establish an improvement for an inequality of Dejun Zhao for the Landau constants.

2000 Mathematics Subject Classification: 26D20, 33B15.

Key words and phrases: Landau's constants, Riemann-Zeta function.

1 Introduction

The Landau's constants are defined by

$$G_n = \sum_{k=0}^{n} \frac{1}{16^k} \left(\begin{array}{c} 2k \\ k \end{array} \right)^2, \ n \geq 0$$

and play an important role in complex analysis.

[1] *Received 2 December, 2009*

Accepted for publication (in revised form) 6 January, 2010

D. Zhao [5] proved the following several sharp inequalities for G_n :

(1)
$$\frac{1}{\pi}\ln(n+1)+c_0-\frac{1}{4\pi(n+1)}+\frac{5}{192\pi(n+1)^2}<G_n \leq \frac{1}{\pi}\ln(n+1)+c_0-\frac{1}{4\pi(n+1)}+\frac{5}{192\pi(n+1)^2}+\frac{3}{128\pi(n+1)^3},$$

where $c_0=\frac{1}{\pi}(\gamma+\ln 16)$, and obtained a new asymptotic formula for G_n

(2)
$$G_n=\frac{1}{\pi}\ln(n+1)+c_0-\frac{1}{4\pi(n+1)}+\frac{5}{192\pi(n+1)^2}+\mathcal{O}\left(\frac{1}{(n+1)^3}\right).$$

In this paper we will establish an improvement for the left-hand inequality of Zhao.

Theorem 1 *We have for all integer $n \geq 1$*

(3)
$$\frac{1}{\pi}\ln(n+1)+c_0-\frac{1}{4\pi(n+1)}+\frac{5}{192\pi(n+1)^2}+\frac{17}{256\pi}\left(\zeta(4)-\sum_{k=1}^{n+1}\frac{1}{k^4}\right)<G_n,$$

where $\zeta(s)=\sum_{k=1}^{\infty}k^{-s}$ is the Riemann-Zeta function.

2 The proof of the Theorem 1

Using [4] and [5] we have

$$\pi p_n^2<\frac{8n+3}{8n^2+5n+1},$$

where $p_n=\frac{(2n)!}{4^n(n!)^2}$. Hence

$$\pi(G_n-G_{n-1})<\frac{8n+3}{8n^2+5n+1}.$$

Let now

$$x_n = G_n - \frac{1}{\pi}\ln(n+1) - c_0 + \frac{1}{4\pi(n+1)} - \frac{5}{192\pi(n+1)^2} - \frac{A}{\pi}\left(\zeta(4) - \sum_{k=1}^{n+1}\frac{1}{k^4}\right),$$

where A is an undetermined constant.

We get for $n \geq 1$

$$\begin{aligned}\pi(x_n - x_{n-1}) &= \pi p_n^2 - \ln\left(1+\frac{1}{n}\right) + \frac{1}{4}\left(\frac{1}{n+1} - \frac{1}{n}\right)\\ &- \frac{5}{192\pi}\left(\frac{1}{(n+1)^2} - \frac{1}{n^2}\right) + \frac{A}{(n+1)^4}\end{aligned}$$

or

$$\begin{aligned}\pi(x_n - x_{n-1}) &< \frac{8n+3}{8n^2+5n+1} - \ln\left(1+\frac{1}{n}\right) + \frac{1}{4}\left(\frac{1}{n+1} - \frac{1}{n}\right)\\ &- \frac{5}{192\pi}\left(\frac{1}{(n+1)^2} - \frac{1}{n^2}\right) + \frac{A}{(n+1)^4}.\end{aligned}$$

Next we consider the function $f(x)$ defined by

$$\begin{aligned}f(x) &= \frac{8x+3}{8x^2+5x+1} - \ln\left(1+\frac{1}{x}\right) + \frac{1}{4}\left(\frac{1}{x+1} - \frac{1}{x}\right)\\ &- \frac{5}{192\pi}\left(\frac{1}{(x+1)^2} - \frac{1}{x^2}\right) + \frac{A}{(x+1)^4}\end{aligned}$$

and we have

$$f'(x) = \frac{Q(x)}{4x^3(x+1)^3(8x^2+5x+1)^2} - \frac{4A}{(x+1)^5},$$

where

$$Q(x) = 68x^5 + \frac{2545}{24}x^4 + \frac{1067}{24}x^3 + \frac{19}{12}x^2 - \frac{41}{24}x - \frac{5}{24}.$$

Now we denote

$$f'(x) = \frac{\mathcal{H}(x)}{4x^3(x+1)^5(8x^2+5x+1)^2}$$

where

$$\begin{aligned}\mathcal{H}(x) &= (68-1024A)x^7+\left(\frac{5809}{24}-1280A\right)x^6\\ &+ \left(\frac{7789}{24}-656A\right)x^5+\left(\frac{4717}{24}-160A\right)x^4\\ &+ \left(\frac{1102}{24}-16A\right)x^3-\frac{49}{24}x^2-\frac{51}{24}x-\frac{5}{24}.\end{aligned}$$

For $A=\dfrac{17}{256}$ we obtain

$$\begin{aligned}\mathcal{H}(x) &= \frac{3769}{24}x^6+\frac{13487}{48}x^5+\frac{2231}{12}x^4\\ &+ \frac{2153}{48}x^3-\frac{49}{29}x^2-\frac{51}{24}x-\frac{5}{24}>0,\ \text{for } x\geq 1.\end{aligned}$$

Hence $f'(x)>0$ for $x\geq 0$ and $\lim_{x\to\infty} f(x)=0$. It implies $f(x)<0$ for $x\geq 1$. Using the Watson asymptotic formula (see [3]) we have $x_n\to 0$. In conclusion $x_n\to 0$, $x_n>0$ for $n\to\infty$ and the inequality of the Theorem 1 holds.

References

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