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A general class of holomorphic functions defined by integral operator [1]

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Abstract

By using the integral operator $I^m f(z)$, $z \in U$ we introduce a class of holomorphic functions, denoted by $\mathcal{I}_n^m(\alpha)$, and we obtain inclusion relations related to this class and some differential subordinations.

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1 Introduction and preliminaries

We denote the complex plane by $\mathbb{C}$ and the open unit disk by U:

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{H}(U)$ be the set of holomorphic function in U.

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For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \left\{f \in \mathcal{H}(U),\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots,\ z \in U\right\}$$

with $\mathcal{H}_0 = \mathcal{H}[0,1]$.

We define the class of normalized analytic functions A_n as

$$A_n = \{f \in \mathcal{H}[U],\ f(z) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}$$

with $A_1 = A$.

Let

$$K = \left\{f \in A : \mathrm{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > 0,\ z \in U\right\}$$

denote the class of normalized convex functions in U.

Let f and g be analytic functions in U. The function f is said to be subordinate to g written $f \prec g$, or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 1 *Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination*

(1) $$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z),$$

then p is called a solution of the differential subordination.The univalent function q is called a dominant of the solutions of the differential subordination, if $p \prec q$ for all p satisfying (1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec p$ for all dominants q of (1) is said to be the best dominant of (1).

Note that the best dominant is unique up a rotation of U.

We will need the following lemma, which is due to D.J. Hallenbeck and St. Ruscheweyh.

Lemma 1 *[2] Let h be a convex function with $h(0) \equiv a$ and let $\gamma \in \mathbb{C}^*$ be a complex number with* $\mathrm{Re}\,\gamma \geq 0$. *If $p \in \mathcal{H}(U)$ with $p(0) = a$ and*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z)$$

then

$$p(z) \prec g(z) \prec h(z)$$

where

$$g(z) = \frac{\gamma}{n z^{\frac{\gamma}{n}-1}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt.$$

The function g is convex and is the best dominant.

The following lemma is due to S.S. Miller and P.T. Mocanu.

Lemma 2 *[4] Let g be a convex function in U and let*

$$h(z) = g(z) + n\alpha z g'(z)$$

where $\alpha > 0$ and n is a positive integer. If $p(z) = g(0) + p_n z^n + \cdots$ is holomorphic in U and

$$p(z) + \alpha z p'(z) \prec h(z),$$

then

$$p(z) \prec g(z)$$

and this result is sharp.

Definition 2 *[6] For $f \in \mathcal{H}(U)$, $f(0) = 0$ and $m \in \mathbb{N}$ we define the operator $I^m f$ by*

$$I^0 f(z) = f(z)$$
$$I^1 f(z) = I f(z) = \int_0^z f(t) t^{-1} dt$$
$$I^m f(z) = I\left(I^{m-1} f(z)\right), \; z \in U.$$

Remark 1 *If* $f \in \mathcal{H}(U)$ *and* $f(z) = \sum_{j=1}^{\infty} a_j z^j$ *then* $I^m f(z) = \sum_{j=1}^{\infty} j^{-m} a_j z^j$.

Remark 2 *For* $m = 1$, $I^m f$ *is the Alexander operator.*

Remark 3 *If we denote* $l(z) = -\log(1-z)$, *then*

$$I^m f(z) = [(l * ... * l) * f]\,(z), f \in \mathcal{H}(U),\ f(0) = 0.$$

By " $*$ " *we denote the Hadamard product or convolution (i.e. if* $f(z) = \sum_{j=0}^{\infty} a_j z^j$, $g(z) = \sum_{j=0}^{\infty} b_j z^j$ *then* $(f * g)\,(z) = \sum_{j=0}^{\infty} a_j b_j z^j$*).*

Remark 4 $I^m f\,(z) = \int_0^z \int_0^{t_m} ... \int_0^{t_2} \dfrac{f\,(t_1)}{t_1 t_2 ... t_m} dt_1 dt_2 ... dt_m$, $f \in \mathcal{H}(U)$, $f(0) = 0$.

Remark 5 $D^m I^m f(z) = I^m D^m f(z) = f(z)$, $f \in \mathcal{H}(U)$, $f(0) = 0$, *where* D^m *is the Sălăgean differential operator.*

2 Main results

Definition 3 *If* $0 \le \alpha < 1$ *and* $m \in \mathbb{N}$, *let* $\mathcal{I}_n^m(\alpha)$ *denote the class of functions* $f \in A_n$ *which satisfy the inequality:*

$$\mathrm{Re}\,[I^m f(z)]' > \alpha. \tag{2}$$

Remark 6 *For* $m = 0$, *we obtain*

$$\mathrm{Re}\, f'(z) > \alpha,\ z \in U. \tag{3}$$

Theorem 1 *If* $0 \le \alpha < 1$ *and* $m, n \in \mathbb{N}$, *then we have*

$$\mathcal{I}_n^m(\alpha) \subset \mathcal{I}_n^{m+1}(\delta), \tag{4}$$

where

$$\delta(\alpha, n) = 2\alpha - 1 + 2(1-\alpha)\frac{1}{n}\beta(\frac{1}{n})$$

and

$$\beta(x) = \int_0^z \frac{t^{x-1}}{1+t}dt.$$

The result is sharp.

Proof. Assume that $f \in \mathcal{I}_n^m(\alpha)$. Then we have

$$I^m f(z) = z[I^{m+1}f(z)]', \; z \in U$$

and differentiating this equality we obtain

(5) $$[I^m f(z)]' = \left[I^{m+1}f(z)\right]' + z\left[I^{m+1}f(z)\right]'', \; z \in U.$$

If $p(z) = [I^{m+1}f(z)]'$, then (5) becomes

(6) $$[I^m f(z)]' = p(z) + zp'(z), \; z \in U.$$

Since $f \in \mathcal{I}_n^m(\alpha)$, from Definition 3 we have

$$\mathrm{Re}[p(z) + zp'(z)] > \alpha, \; z \in U$$

which is equivalent to

$$p(z) + zp'(z) \prec \frac{1 + (2\alpha - 1)z}{1+z} \equiv h(z), \; z \in U.$$

Therefore, from Lemma 1 results that

$$p(z) \prec g(z) \prec h(z), \; z \in U$$

where

$$g(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z \frac{1 + (2\alpha - 1)t}{1+t} t^{\frac{1}{n}-1} dt$$

$$= \frac{1}{nz^{\frac{1}{n}}}\int_0^z (2\alpha-1)t^{\frac{1}{n}-1}dt + \frac{2(1-\alpha)}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}dt$$

$$= 2\alpha - 1 + 2(1-\alpha)\frac{1}{n}\beta\left(\frac{1}{n}\right)\frac{1}{z^{\frac{1}{n}}},\ z \in U.$$

Moreover, the function g is convex and is the best dominant.

From $p(z) \prec g(z)$, it results that

$$\operatorname{Re} p(z) > \operatorname{Re} g(1) = \delta(\alpha, n) = 2\alpha - 1 + 2(1-\alpha)\frac{1}{n}\beta\left(\frac{1}{n}\right),$$

from which we deduce that $\mathcal{I}_n^m(\alpha) \subset \mathcal{I}_n^{m+1}(\delta)$.

Theorem 2 *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + nzg'(z),\ z \in U.$$

If $f \in A_n$ and verifies the differential subordination

$$[I^m f(z)]' \prec h(z) \tag{7}$$

then

$$\left[I^{m+1} f(z)\right]' \prec g(z),\ z \in U$$

and this result is sharp.

Proof. From the relation (6) and differential subordination (7), we obtain

$$p(z) + zp'(z) \prec g(z) + nzg'(z) \equiv h(z).$$

By using Lemma 2, we have

$$p(z) \prec g(z)$$

i.e.

$$\left[I^{m+1} f(z)\right]' \prec g(z)$$

and this result is sharp.

Theorem 3 *Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, $h'(0) \neq 0$, which verifies the inequality*

$$\text{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2n},\ z \in U.$$

If $f \in A_n$ and verifies the differential subordination

$$[I^m f(z)]' \prec h(z), \tag{8}$$

then

$$\left[I^{m+1} f(z)\right]' \prec g(z),\ z \in U$$

where

$$g(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt,\ z \in U.$$

The function g is convex and is the best dominant.

Proof. A simple application of the differential subordination technique shows that the function g is convex. From

$$I^m f(z) = z[I^{m+1} f(z)]'$$

we obtain

$$[I^m f(z)]' = \left[I^{m+1} f(z)\right]' + z\left[I^{m+1} f(z)\right]'',\ z \in U.$$

If we assume

$$p(z) = [I^{m+1} f(z)]'$$

then

$$[I^m f(z)]' = p(z) + zp'(z),\ z \in U$$

hence (8) becomes

$$p(z) + zp'(z) \prec h(z).$$

Moreover, from Lemma 1 it results that

$$p(z) \prec g(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt$$

i.e.

$$\left[I^{m+1} f(z)\right]' \prec g(z)$$

and g is the best dominant.

Theorem 4 *Let g be a convex function, $g(0) = 1$, and*

$$h(z) = g(z) + nzg'(z).$$

If $f \in A_n$ and verifies the differential subordination

$$[I^m f(z)]' \prec h(z) \tag{9}$$

then

$$\frac{I^m f(z)}{z} \prec g(z), \ z \in U, \ z \neq 0.$$

The result is sharp.

Proof. If

$$p(z) = \frac{I^m f(z)}{z}, \ z \in U, \ z \neq 0$$

then it results that

$$I^m f(z) = zp(z). \tag{10}$$

Differentiating (10), we obtain

$$[I^m f(z)]' = p(z) + zp'(z), \ z \in U.$$

hence (9) becomes

$$p(z) + zp'(z) \prec h(z) \equiv g(z) + nzg'(z), \ z \in U. \tag{11}$$

Therefore, from Lemma 2 it results that

$$p(z) \prec g(z),$$

i.e.

$$\frac{I^m f(z)}{z} \prec g(z), \; z \in U$$

and the result is sharp.

Theorem 5 *Let* $f \in \mathcal{H}(U)$, $h(0) = 0$, $h'(0) \neq 0$ *which satisfy the inequality*

$$\text{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \; z \in U.$$

If $f \in A_n$ *and verifies the differential subordination*

$$[I^m f(z)]' \prec h(z) \tag{12}$$

then

$$\frac{I^m f(z)}{z} \prec g(z), \; z \in U, \; z \neq 0,$$

where

$$g(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt, \; z \in U.$$

The function g *is convex and is the best dominant.*

Proof. A simple application of the differential subordination technique shows that the function g is convex.

Differentiating (10) we obtain

$$[I^m f(z)]' = p(z) + zp'(z).$$

Then (12) becomes

$$p(z) + zp'(z) \prec h(z), \; z \in U.$$

By using Lemma 1 we have

$$p(z) \prec g(z)$$

where

$$g(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt$$

and g is convex and is the best dominant.

For $n = 1$, this results was obtained in [1].

We remark that in the case of Sălăgean differential operator a similar results was obtained by G.I. Oros in [5].

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