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A new univalent integral operator defined by Al-Oboudi differential operator [1]

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Abstract

In [3], Breaz and Breaz gave an univalence condition of the integral operator $G_{n,\alpha}$ introduced in [2]. The purpose of this paper is to give univalence condition of the generalized integral operator $G_{n,m,\alpha}$ defined in [4]. Our results generalize the results of [3].

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1 Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

(1) $$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

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which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and

$$\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{U}\}.$$

For $f \in \mathcal{A}$, Al-Oboudi [1] introduced the following operator:

$$D^0 f(z) = f(z), \tag{2}$$

$$D^1 f(z) = (1-\delta)f(z) + \delta z f'(z) = D_\delta f(z), \quad \delta \geq 0 \tag{3}$$

$$D^n f(z) = D_\delta(D^{n-1} f(z)), \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{4}$$

If f is given by (1), then from (3) and (4) we see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k z^k, \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{5}$$

with $D^n f(0) = 0$.

Remark 1 *When $\delta = 1$, we get Sălăgean's differential operator [9].*

The following results will be required in our investigation.

General Schwarz Lemma. [5] *Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m \quad (z \in \mathbb{U}_R).$$

The equality can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

Theorem A. [7] *Let α be a complex number with $Re\alpha > 0$ and $f \in \mathcal{A}$. If f satisfies*

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \quad (z \in \mathbb{U}),$$

then, for any complex number β with $Re\beta \geq Re\alpha$, the integral operator

$$F_\beta(z) = \left\{\beta \int_0^z t^{\beta-1} f'(t)dt\right\}^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Theorem B. [6] *Let $f \in \mathcal{A}$ satisfy the following inequality:*

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| \leq 1 \quad (z \in \mathbb{U}). \tag{6}$$

Then f is univalent in $\mathbb{U}$.

Theorem C. [8] *Assume that $g \in \mathcal{A}$ satisfies condition* (6), *and let α be a complex number with*

$$|\alpha - 1| \leq \frac{Re\alpha}{3}.$$

If

$$|g(z)| \leq 1, \quad \forall z \in \mathbb{U}$$

then the function

$$G_\alpha(z) = \left\{\alpha \int_0^z (g(t))^{\alpha-1}\, dt\right\}^{\frac{1}{\alpha}} \tag{7}$$

is of class $\mathcal{S}$.

In [2], Breaz and Breaz considered the integral operator

$$G_{n,\alpha}(z) := \left\{[n(\alpha-1)+1] \int_0^z (g_1(t))^{\alpha-1} \cdots (g_n(t))^{\alpha-1}\, dt\right\}^{\frac{1}{n(\alpha-1)+1}}, \tag{8}$$

$(g_1, \ldots, g_n \in \mathcal{A})$, and proved that the function $G_{n,\alpha}$ is univalent in $\mathbb{U}$.

Remark 2 *Note that for $n = 1$, we obtain the integral operator G_α defined by* (7).

Theorem D. [3] *Let* $g_i \in \mathcal{A}, \quad \forall i = 1, \ldots, n,\ n \in \mathbb{N}$, *satisfy the properties*

$$\left| \frac{z^2 g_i'(z)}{(g_i(z))^2} - 1 \right| < 1, \quad \forall z \in \mathbb{U}, \quad \forall i = 1, \ldots, n$$

and $\alpha \in \mathbb{C}$ *with*

$$|\alpha - 1| \leq \frac{Re\alpha}{3n}.$$

If

$$|g_i(z)| \leq 1,\ \forall z \in \mathbb{U},\ \forall i = 1, \ldots, n,$$

then the function $G_{n,\alpha}$ *defined by* (8) *is univalent.*

In [4], the author introduced a new general integral operator by means of the Al-Oboudi differential operator as follows.

Definition 1 [4] *Let* $n \in \mathbb{N}$, $m \in \mathbb{N}_0$ *and* $\alpha \in \mathbb{C}$. *We define the integral operator* $G_{n,m,\alpha}$ *by*

(9)
$$G_{n,m,\alpha}(z) := \left\{ [n(\alpha - 1) + 1] \int_0^z \prod_{j=1}^{n} (D^m g_j(t))^{\alpha - 1}\, dt \right\}^{\frac{1}{n(\alpha-1)+1}} \qquad (z \in \mathbb{U}),$$

where $g_1, \ldots, g_n \in \mathcal{A}$ *and* D^m *is the Al-Oboudi differential operator.*

Remark 3 *In the special case* $n = 1$, *we obtain the integral operator*

$$G_{m,\alpha}(z) := \left\{ \alpha \int_0^z (D^m g(t))^{\alpha - 1}\, dt \right\}^{\frac{1}{\alpha}} \qquad (z \in \mathbb{U}). \tag{10}$$

Remark 4 *If we set* $m = 0$ *in* (9) *and* (10), *then we obtain the integral operators defined in* (8) *and* (7), *respectively.*

2 Main Results

Theorem 1 *Let $M_j \geq 1$, each of the functions $g_j \in \mathcal{A}$ $(j \in \{1, \ldots, n\})$ satisfies the inequality*

$$\left| \frac{z^2 \left(D^m g_j(z)\right)'}{\left(D^m g_j(z)\right)^2} - 1 \right| \leq 1 \quad (z \in \mathbb{U};\ m \in \mathbb{N}_0). \tag{11}$$

and $\alpha \in \mathbb{C}$ with

$$|\alpha - 1| \leq \frac{Re\alpha}{\sum_{j=1}^{n} (2M_j + 1)},\ Re\left(n\left(\alpha - 1\right) + 1\right) \geq Re\alpha > 0.$$

If

$$|D^m g_j(z)| \leq M_j \quad (z \in \mathbb{U};\ j \in \{1, \ldots, n\}),$$

then the integral operator $G_{n,m,\alpha}$ defined by (9) *is in the univalent function class $\mathcal{S}$.*

Proof. Since $g_j \in \mathcal{A}$ $(j \in \{1, \ldots, n\})$, by (5), we have

$$\frac{D^m g_j(z)}{z} = 1 + \sum_{k=2}^{\infty} \left[1 + (k-1)\delta\right]^m a_{k,j} z^{k-1} \quad (m \in \mathbb{N}_0)$$

and

$$\frac{D^m g_j(z)}{z} \neq 0$$

for all $z \in \mathbb{U}$.

Also we note that

$$G_{n,m,\alpha}(z) = \left\{ [n(\alpha - 1) + 1] \int_0^z t^{n(\alpha-1)} \prod_{j=1}^{n} \left(\frac{D^m g_j(t)}{t} \right)^{\alpha - 1} dt \right\}^{\frac{1}{n(\alpha-1)+1}}.$$

Define a function

$$f(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{D^m g_j(t)}{t} \right)^{\alpha - 1} dt.$$

Then we obtain

$$f'(z) = \prod_{j=1}^{n} \left(\frac{D^m g_j(z)}{z} \right)^{\alpha - 1}. \tag{12}$$

It is clear that $f(0) = f'(0) - 1 = 0$.

The equality (12) implies that

$$\ln f'(z) = (\alpha - 1) \sum_{j=1}^{n} \ln \frac{D^m g_j(z)}{z}$$

or equivalently

$$\ln f'(z) = (\alpha - 1) \sum_{j=1}^{n} \left(\ln D^m g_j(z) - \ln z \right).$$

By differentiating above equality, we get

$$\frac{f''(z)}{f'(z)} = (\alpha - 1) \sum_{j=1}^{n} \left(\frac{(D^m g_j(z))'}{D^m g_j(z)} - \frac{1}{z} \right).$$

Hence we obtain

$$\frac{z f''(z)}{f'(z)} = (\alpha - 1) \sum_{j=1}^{n} \left(\frac{z \left(D^m g_j(z)\right)'}{D^m g_j(z)} - 1 \right),$$

which readily shows that

$$\begin{aligned} \frac{1 - |z|^{2Re\alpha}}{Re\alpha} \left| \frac{z f''(z)}{f'(z)} \right| &\leq \frac{1 - |z|^{2Re\alpha}}{Re\alpha} |\alpha - 1| \sum_{j=1}^{n} \left(\left| \frac{z \left(D^m g_j(z)\right)'}{D^m g_j(z)} \right| + 1 \right) \\ &\leq \frac{|\alpha - 1|}{Re\alpha} \sum_{j=1}^{n} \left(\left| \frac{z^2 \left(D^m g_j(z)\right)'}{\left(D^m g_j(z)\right)^2} \right| \left| \frac{D^m g_j(z)}{z} \right| + 1 \right). \end{aligned}$$

From the hypothesis, we have $|g_j(z)| \leq M_j$ $(j \in \{1, \ldots, n\}\ ;\ z \in \mathbb{U})$, then by the General Schwarz Lemma, we obtain that

$$|g_j(z)| \leq M_j\, |z| \quad (j \in \{1, \ldots, n\}\ ;\ z \in \mathbb{U}).$$

Then we find

$$\begin{aligned}\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| &\le \frac{|\alpha-1|}{Re\alpha}\sum_{j=1}^{n}\left(\left|\frac{z^2\left(D^m g_j(z)\right)'}{\left(D^m g_j(z)\right)^2}-1\right|M_j+M_j+1\right)\\ &\le \frac{|\alpha-1|}{Re\alpha}\sum_{j=1}^{n}(2M_j+1)\le 1\end{aligned}$$

since $|\alpha-1|\le\frac{Re\alpha}{\sum_{j=1}^{n}(2M_j+1)}$. Applying Theorem A, we obtain that $G_{n,m,\alpha}$ is in the univalent function class $\mathcal{S}$.

Corollary 1 *Let* $M\ge 1$*, each of the functions* $g_j\in\mathcal{A}$ $(j\in\{1,\dots,n\})$ *satisfies the inequality* (11) *and* $\alpha\in\mathbb{C}$ *with*

$$|\alpha-1|\le\frac{Re\alpha}{(2M+1)\,n},\ Re\left(n\left(\alpha-1\right)+1\right)\ge Re\alpha>0.$$

If

$$\left|D^m g_j(z)\right|\le M\quad(z\in\mathbb{U};\ j\in\{1,\dots,n\}),$$

then the integral operator $G_{n,m,\alpha}$ *defined by* (9) *is in the univalent function class* $\mathcal{S}$*.*

Proof. In Theorem 1, we consider $M_1=\cdots=M_n=M$.

Corollary 2 *Let each of the functions* $g_j\in\mathcal{A}$ $(j\in\{1,\dots,n\})$ *satisfies the inequality* (11) *and* $\alpha\in\mathbb{C}$ *with*

$$|\alpha-1|\le\frac{Re\alpha}{3n},\ Re\left(n\left(\alpha-1\right)+1\right)\ge Re\alpha>0.$$

If

$$\left|D^m g_j(z)\right|\le 1\quad(z\in\mathbb{U};\ j\in\{1,\dots,n\}),$$

then the integral operator $G_{n,m,\alpha}$ *defined by* (9) *is in the univalent function class* $\mathcal{S}$*.*

Proof. In Corollary 1, we consider $M = 1$.

Remark 5 *If we set $m = 0$ in Corollary 2, then we have Theorem D.*

Corollary 3 *Let the function $g \in \mathcal{A}$ satisfies the inequality (11) and $\alpha \in \mathbb{C}$ with*

$$|\alpha - 1| \leq \frac{Re\alpha}{3}, \ Re\alpha > 0.$$

If

$$|D^m g(z)| \leq 1 \quad (z \in \mathbb{U}),$$

then the integral operator $G_{m,\alpha}$ defined by (10) is in the univalent function class $\mathcal{S}$.

Proof. In Corollary 2, we consider $n = 1$.

Remark 6 *If we set $m = 0$ in Corollary 3, then we have Theorem C.*

References

[1] F. M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Int. J. Math. Math. Sci. 2004, no. 25-28, 1429-1436.

[2] D. Breaz and N. Breaz, *Univalence of an integral operator*, Mathematica (Cluj) **47** (2005), no. 70, 35-38.

[3] D. Breaz and N. Breaz, *An integral univalent operator*, Acta Math. Univ. Comenian. (N.S.) **76** (2007), no. 2, 137-142.

[4] S. Bulut, *Univalence condition for a new generalization of the family of integral operators*, Acta Univ. Apulensis Math. Inform. No. **18** (2009), 71-78.

[5] Z. Nehari, Conformal Mapping, Dover, New York, NY, USA, 1975.

[6] S. Ozaki and M. Nunokawa, *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc. **33** (1972), 392-394.

[7] N. Pascu, *An improvement of Becker's univalence criterion*, Proceedings of the Commemorative Session Simion Stoilow, Brasov, (1987), 43-48.

[8] V. Pescar, *New criteria for univalence of certain integral operators*, Demonstratio Math. **33** (2000), no. 1, 51-54.

[9] G. Ş. Sălăgean, *Subclasses of univalent functions*, Complex Analysis-Fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., vol. 1013, Springer, Berlin, 1983, pp. 362-372.

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