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A common fixed point theorem for weakly compatible mappings in fuzzy metric spaces [1]

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Abstract

In this paper, we prove a common fixed point theorem for weakly compatible mappings in fuzzy metric spaces using the property (E.A).

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1 Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [15] in 1965. To use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. George and Veeramani [7] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [10] and defined the Hausdorff topology of fuzzy metric spaces which have very important applications in quantum particle physics particularly in connections with both string and $E-$infinity theory which were given and studied by El- Naschie [2, 3, 4, 5, 6] and [13]. They showed also that every metric induces a fuzzy metric. Vasuki [14] obtained the fuzzy version of common fixed point theorem which had extra conditions, in fact, he proved a fuzzy common fixed point theorem by a strong definition of Cauchy sequence, see [7]. First, we give some definitions.

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Definition 1 ([12]) *A binary operation* $* : [0,1]^2 \longrightarrow [0,1]$ *is called a continuous t-norm if* $([0,1], *)$ *is an abelian topological monoid; i.e.,*

(1) $*$ *is associative and commutative,*
(2) $*$ *is continuous,*
(3) $a * 1 = a$ *for all* $a \in [0,1]$,
(4) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *for each* $a, b, c, d \in [0,1]$.

Two typical examples of a continuous $t-$norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2 ([7]) *The 3-tuple* $(X, M, *)$ *is called a fuzzy metric space if* X *is an arbitrary non-empty set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times [0, \infty)$ *satisfying the following conditions for each* $x, y, z \in X$ *and* $t, s > 0$,

(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ *if and only if* $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ *is continuous.*

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with a center $x \in X$ and a radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable.

Example 1 *Let* $X = \mathbb{R}$. *Denote* $a * b = a.b$ *for all* $a, b \in [0, 1]$. *For each* $t \in (0, \infty)$, *define*

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

for all $x, y \in X$.

Example 2 *Let* X *be an arbitrary non-empty set and* ψ *be an increasing and a continuous function of* $\mathbb{R}_+$ *into* $(0, 1)$ *such that* $\lim_{t \longrightarrow \infty} \psi(t) = 1$. *Three* *typical examples of these functions are* $\psi(x) = \dfrac{x}{x+1}$, $\psi(x) = \sin(\dfrac{\pi x}{2x+1})$ *and* $\psi(x) = 1 - e^{-x}$*. Denote* $a * b = a.b$ *for all* $a, b \in [0,1]$*. For each* $t \in (0, \infty)$*, define*

$$M(x, y, t) = \psi(t)^{d(x,y)}$$

for all $x, y \in X$*, where* $d(x, y)$ *is an ordinary metric. It is easy to see that* $(X, M, *)$ *is a fuzzy metric space.*

Definition 3 ([7]) *Let* $(X, M, *)$ *be a fuzzy metric space.*

(i) A sequence $\{x_n\}$ *in* X *is said to be convergent to* $x \in X$ *if for each* $\epsilon > 0$ *and each* $t > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that* $M(x_n, x, t) > 1 - \epsilon$ *for all* $n \geq n_0$*; i.e.,* $M(x_n, x, t) \to 1$ *as* $n \to \infty$ *for all* $t > 0$*.*

(ii) A sequence $\{x_n\}$ *in* X *is said to be Cauchy if for each* $\epsilon > 0$ *and each* $t > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that* $M(x_n, x_m, t) > 1 - \epsilon$ *for all* $n, m \geq n_0$*; i.e.,* $M(x_n, x_m, t) \to 1$ *as* $n, m \to \infty$ *for all* $t > 0$*.*

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Lemma 1 ([8]) *For all* $x, y \in X$*,* $M(x, y, .)$ *is a non-decreasing function.*

Definition 4 *Let* $(X, M, *)$ *be a fuzzy metric space.* M *is said to be continuous on* $X^2 \times (0, \infty)$ *if*

$$\lim_{n \to \infty} M(x_n, y_n, t_n) = M(x, y, t),$$

whenever $\{(x_n, y_n, t_n)\}$ *is a sequence in* $X^2 \times (0, \infty)$ *which converges to a point* $(x, y, t) \in X^2 \times (0, \infty)$*; i.e.,*

$$\lim_{n \to \infty} M(x_n, x, t) = \lim_{n \to \infty} M(y_n, y, t) = 1 \text{ and } \lim_{n \to \infty} M(x, y, t_n) = M(x, y, t)$$

Lemma 2 ([8]) M *is a continuous function on* $X^2 \times (0, \infty)$*.*

Let A and S be self-mappings of a fuzzy metric space $(X, M, *)$.

Definition 5 ([9]) A *and* S *are said to be weakly compatible if they commute at their coincidence points; i.e,* $Ax = Sx$ *for some* $x \in X$ *implies that* $ASx = SAx$*.*

Definition 6 ([1]) *The pair (A, S) satisfies the property (E.A) if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} M(Ax_n, u, t) = \lim_{n\to\infty} M(Sx_n, u, t) = 1$$

for some $u \in X$ and all $t > 0$.

Example 3 *Let $X = \mathbb{R}$ and $M(x, y, t) = \dfrac{t}{t + |x - y|}$ for every $x, y \in X$ and $t > 0$. Define A and S by $Ax = 2x + 1$, $Sx = x + 2$ and the sequence $\{x_n\}$ by $x_n = 1 + \dfrac{1}{n}$, $n = 1, 2, \ldots.$ We have*

$$\lim_{n\to\infty} M(Ax_n, 3, t) = \lim_{n\to\infty} M(Sx_n, 3, t) = 1$$

for every $t > 0$. Then, the pair (A, S) satisfies the property (E.A). However, A and S are not weakly compatible.

The following example shows that there are some pairs of mappings which do not satisfy the property (E.A).

Example 4 *Let $X = \mathbb{R}$ and $M(x, y, t) = \dfrac{t}{t + |x - y|}$ for every $x, y \in X$ and $t > 0$. Define A and B by $Ax = x + 1$ and $Sx = x + 2$. Assume that there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} M(Ax_n, u, t) = \lim_{n\to\infty} M(Sx_n, u, t) = 1$$

for some $u \in X$ and all $t > 0$. Therefore

$$\lim_{n\to\infty} M(x_n + 1, u, t) = \lim_{n\to\infty} M(x_n + 2, u, t) = 1.$$

We conclude that $x_n \to u - 1$ and $x_n \to u - 2$ which is a contradiction. Hence, the pair (A, S) do not satisfy property (E.A).

It is our purpose in this paper to prove a common fixed point theorem for weakly compatible mappings satisfying a contractive condition in fuzzy metric spaces using the property (E.A).

2 Main Results

Let Φ be the set of all increasing and continuous functions $\phi : (0,1] \longrightarrow (0,1]$, such that $\phi(t) > t$ for every $t \in (0,1)$.

Example 5 *Let* $\phi : (0,1] \longrightarrow (0,1]$ *defined by* $\phi(t) = t^{1/2}$.

Theorem 1 *Let* $(X, M, *)$ *be a fuzzy metric space and* S *and* T *be self-mappings of* X *satisfying the following conditions:*
(i) $T(X) \subseteq S(X)$ *and* $T(X)$ *or* $S(X)$ *is a closed subset of* X,
(ii)

$$M(Tx,Ty,t) \geq \phi(\min \left\{ \begin{array}{l} M(Sx,Sy,t), \\ \sup_{t_1+t_2=\frac{2}{k}t} \min \left\{ \begin{array}{l} M(Sx,Tx,t_1), \\ M(Sy,Ty,t_2) \end{array} \right\}, \\ \sup_{t_3+t_4=\frac{2}{k}t} \max \left\{ \begin{array}{l} M(Sx,Ty,t_3), \\ M(Sy,Tx,t_4) \end{array} \right\} \\ \end{array} \right\})$$

for all $x, y \in X$, $t > 0$ *and for some* $1 \leq k < 2$. *Suppose that the pair* (T, S) *satisfies the property (E.A) and* (T, S) *is weakly compatible. Then* S *and* T *have a unique common fixed point in* X.

Proof. Since the pair (T, S) satisfies the property (E.A), there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} M(Tx_n, z, t) = \lim_{n\to\infty} M(Sx_n, z, t) = 1$$

for some $z \in X$ and every $t > 0$. Suppose that $S(X)$ is a closed subset of X. Then, there exists $v \in X$ such that $Sv = z$ and so

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = Sv = z. \qquad (*)$$

Assume that $T(X)$ is a closed subset of X. Therefore, there exists $v \in X$ such that $Sv = z$. Hence $(*)$ still holds. Now, we show that $Tv = Sv$. Suppose that $Tv \neq Sv$. It is not difficult to prove that there exists $t_0 > 0$ such that

$$M(Tv, Sv, \frac{2}{k}t_0) > M(Tv, Sv, t_0). \qquad (**)$$

If not, we have $M(Tv, Sv, t) = M(Tv, Sv, \frac{2}{k}t)$ for all $t > 0$. Repeatedly using this equality, we obtain

$$M(Tv, Sv, t) = M(Tv, Sv, \frac{2}{k}t) = \cdots = M(Tv, Sv, (\frac{2}{k})^n t) \longrightarrow 1 \quad (n \longrightarrow \infty).$$

This shows that $M(Tv, Sv, t) = 1$ for all $t > 0$ which contradicts $Tv \neq Sv$ and so $(**)$ is proved.

Using (ii) we get

$$\begin{aligned} M(Tx_n, Tv, t_0) &\geq \phi(\min \left\{ \begin{array}{l} M(Sx_n, Sv, t_0), \\ \sup_{t_1+t_2=\frac{2}{k}t_0} \min \left\{ \begin{array}{l} M(Sx_n, Tx_n, t_1), \\ M(Sv, Tv, t_2) \end{array} \right\}, \\ \sup_{t_3+t_4=\frac{2}{k}t_0} \max \left\{ \begin{array}{l} M(Sx_n, Tv, t_3), \\ M(Sv, Tx_n, t_4) \end{array} \right\} \end{array} \right\}) \\ &\geq \phi(\min \left\{ \begin{array}{l} M(Sx_n, Sv, t_0), \\ \min \left\{ M(Sx_n, Tx_n, \epsilon), M(Sv, Tv, \frac{2}{k}t_0 - \epsilon) \right\}, \\ \max \left\{ M(Sx_n, Tv, \frac{2}{k}t_0 - \epsilon), M(Sv, Tx_n, \epsilon) \right\} \end{array} \right\}) \end{aligned}$$

$\forall \epsilon \in (0, \frac{2}{k}t_0)$. As $n \to \infty$, it follows that

$$\begin{aligned} M(Sv, Tv, t_0) &\geq \phi(\min \left\{ \begin{array}{l} M(Sv, Sv, t_0), \\ \min \left\{ \begin{array}{l} M(Sv, Sv, \epsilon), \\ M(Sv, Tv, \frac{2}{k}t_0 - \epsilon) \end{array} \right\}, \\ \max \left\{ \begin{array}{l} M(Sv, Tv, \frac{2}{k}t_0 - \epsilon), \\ M(Sv, Sv, \epsilon) \end{array} \right\} \end{array} \right\}) \\ &= \phi(M(Sv, Tv, \frac{2}{k}t_0 - \epsilon)) \\ &> M(Sv, Tv, \frac{2}{k}t_0 - \epsilon) \end{aligned}$$

as $\epsilon \longrightarrow 0$, we have

$$M(Sv, Tv, t_0) \geq M(Sv, Tv, \frac{2}{k}t_0)$$

which is a contradiction. Therefore, $z = Sv = Tv$. Since S and T are weakly compatible, we have $Tz = Sz$.

Now, we show that z is a common fixed point of S and T. If $Tz \neq z$ using (ii) we obtain

$$\begin{aligned} M(z,Tz,t) &\geq \phi(\min\left\{\begin{array}{l} M(z,Tz,t), \\ \sup_{t_1+t_2=\frac{2}{k}t}\min\left\{\begin{array}{l} M(z,Tz,t_1), \\ M(Sz,Tz,t_2) \end{array}\right\}, \\ \sup_{t_3+t_4=\frac{2}{k}t}\max\left\{\begin{array}{l} M(z,Tz,t_3), \\ M(Tz,z,t_4) \end{array}\right\} \\ \end{array}\right\}) \\ &\geq \phi(\min\left\{\begin{array}{l} M(z,Tz,t), \\ \min\left\{\begin{array}{l} M(z,Tz,\frac{2}{k}t-\epsilon), \\ M(Sz,Tz,\epsilon) \end{array}\right\}, \\ \max\left\{\begin{array}{l} M(z,Tz,\frac{2}{k}t-\epsilon), \\ M(Tz,z,\epsilon) \end{array}\right\} \\ \end{array}\right\}) \end{aligned}$$

for all $\epsilon \in (0, \frac{2}{k}t)$. As $\epsilon \longrightarrow 0$, we have

$$\begin{aligned} M(z,Tz,t) &\geq \phi(\min\{M(z,Tz,t), M(z,Tz,\frac{2}{k}t)\}) \\ &= \phi(M(z,Tz,t)) > M(z,Tz,t) \end{aligned}$$

which is a contradiction. Hence $Tz = Sz = z$. Thus z is a common fixed point of S and T. The uniqueness of z follows from the inequality (ii).

Example 6 *Let $(X, M, *)$ be a fuzzy metric space, where $X = [0,1]$ with a t-norm defined $a * b = a.b$ for all $a, b \in [0,1]$ and ψ is an increasing and a continuous function of $\mathbb{R}_+$ into $(0,1)$ such $\lim_{t \longrightarrow \infty} \psi(t) = 1$. For each $t \in (0,\infty)$, define*

$$M(x,y,t) = \psi(t)^{|x-y|}$$

for all $x, y \in X$, see example 2. Define self-maps T and S on X as follows:

$$Tx = \frac{x+2}{3}, \qquad Sx = \tan(\frac{\pi x}{4})$$

It is easy to see that
(i) $T(X) = [\frac{2}{3}, 1] \subseteq [0,1] = S(X)$,
(ii) *For a sequence* $x_n = 1 - \frac{1}{n}$, *we have*

$$\lim_{n\to\infty} M(Tx_n, 1, t) = \psi(t)^{|\frac{1-1/n+2}{3}-1|} = \lim_{n\to\infty} M(Sx_n, 1, t) = \psi(t)^{|\tan(\frac{\pi(1-1/n)}{4})-1|} = 1$$

for every $t > 0$. *Hence the pair* (T, S) *satisfies the property (E.A). It is easy to see that the pair* (T, S) *is weakly compatible. Let* $\phi : (0,1] \longrightarrow (0,1]$ *defined by* $\phi(t) = t^{1/2}$. *As*

$$|\tan(\frac{\pi x}{4}) - \tan(\frac{\pi y}{4})| \geq \frac{\pi}{4}|x-y|$$

we get

$$\begin{aligned} M(Tx, Ty, t) &= \psi(t)^{\frac{1}{3}|x-y|} \\ &\geq \psi(t)^{\frac{\pi}{8}|x-y|} = \phi(M(Sx, Sy, t)). \end{aligned}$$

Thus for $\phi(t) = t^{1/2}$ *we have*

$$M(Tx,Ty,t) \geq \phi(\min \left\{ \begin{array}{l} M(Sx,Sy,t), \\ \sup_{t_1+t_2=\frac{2}{k}t} \min \left\{ \begin{array}{c} M(Sx,Tx,t_1), \\ M(Sy,Ty,t_2) \end{array} \right\}, \\ \sup_{t_3+t_4=\frac{2}{k}t} \max \left\{ \begin{array}{c} M(Sx,Ty,t_3), \\ M(Sy,Tx,t_4) \end{array} \right\} \end{array} \right\})$$

for all $x, y \in X$, $t > 0$ *and for some* $1 \leq k < 2$. *All conditions of Theorem 1 hold and* $z = 1$ *is a unique common fixed point of* S *and* T.

Corollary 1 *Let* T *and* S *be self-mappings of a fuzzy metric space* $(X, M, *)$ *satisfying the following conditions:*

(i) $T^n(X) \subseteq S^m(X)$, $T^n(X)$ *or* $S^m(X)$ *is a closed subset of* X *and* $T^nS = ST^n$, $TS^m = S^mT$,

(ii)

$$M(T^nx,T^ny,t)\geq\phi(\min\left\{\begin{array}{l}M(S^mx,S^my,t),\\ \sup_{t_1+t_2=\frac{2}{k}t}\min\left\{\begin{array}{l}M(S^mx,T^nx,t_1),\\ M(S^my,T^ny,t_2)\end{array}\right\},\\ \sup_{t_3+t_4=\frac{2}{k}t}\max\left\{\begin{array}{l}M(S^mx,T^ny,t_3),\\ M(S^my,T^nx,t_4)\end{array}\right\}\end{array}\right\})$$

for all $x,y\in X$*, for some* $n,m=2,3,\cdots$*,* $t>0$ *and for some* $1\leq k<2$*. Suppose that the pair* (T^n,S^m) *satisfies the property (E.A) and* (T^n,S^m) *is weakly compatible. Then* S *and* T *have a unique common fixed point in* X*.*

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