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Fixed points for occasionally weakly compatible maps [1]

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Abstract

In this article, using occasionally weak compatibility due to Al-Thagafi and Shahzad [1], we generalize some common fixed point theorems of Greguš contraction type in a normed space.

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1 Introduction

Recently, Jungck [5] introduced the notion of weakly compatible maps as follows:

Definition 1 *Self-maps f and g of a metric space $(\mathcal{X}, d)$ are called weakly compatible if $ft = gt$ for some $t \in \mathcal{X}$ implies that $fgt = gft$.*

More recently, Al-Thagafi and Shahzad [1] weakened the weak compatibility by giving the so-called occasionally weak compatibility.

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Definition 2 *Self-maps f and g of a set $\mathcal{X}$ are said to be occasionally weakly compatible if and only if there exists a point $t \in \mathcal{X}$ such that $ft = gt$ and $fgt = gft$.*

In their paper [3], Djoudi and Nisse proved a common fixed point theorem of Greguš contraction type in a Banach space by using the weak compatibility.

Theorem 1 *Let f, g, h and k be maps from a Banach space $\mathcal{X}$ into itself having the conditions*
(1.1) $f(\mathcal{X}) \subset k(\mathcal{X})$ and $g(\mathcal{X}) \subset h(\mathcal{X})$,
(1.2) the inequality

$$\begin{aligned} \|fx - gy\|^p \leq\; & \varphi(a\|hx - ky\|^p + (1-a)\max\{\alpha\|fx - hx\|^p, \\ & \beta\|gy - ky\|^p, \|fx - hx\|^{\frac{p}{2}}\|fx - ky\|^{\frac{p}{2}}, \\ & \|fx - ky\|^{\frac{p}{2}}\|gy - hx\|^{\frac{p}{2}}, \\ & \frac{1}{2}(\|fx - hx\|^p + \|gy - ky\|^p)\}); \end{aligned}$$

for all $x, y \in \mathcal{X}$, where $0 < a \leq 1$, $0 < \alpha, \beta \leq 1$, $p \geq 1$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that φ is upper semi-continuous, nondecreasing and $\varphi(t) < t$ for any $t > 0$,
(1.3) one of $f(\mathcal{X})$ or $g(\mathcal{X})$ is closed.
If the pairs $\{f, h\}$ and $\{g, k\}$ are weakly compatible, then f, g, h and k have a unique common fixed point in $\mathcal{X}$.

In this work, we give some results which include the analogue of certain results in [2], [3], [4], [6], [7], [8] and references therein.

2 Main Results

Theorem 2 *Let f, g, h, k be maps from a normed space $(\mathcal{X}, \|.\|)$ having inequality (1.2) for all $x, y \in \mathcal{X}$, where $0 < a \leq 1$, $\alpha, \beta > 0$, $p \geq 1$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(t) < t$ for any $t > 0$. If pairs of maps $\{f, h\}$ and $\{g, k\}$ are occasionally weakly compatible. Then f, g, h and k have a unique common fixed point.*

Proof. Since pairs of maps $\{f, h\}$ and $\{g, k\}$ are occasionally weakly compatible, then, there exist two elements u and v in $\mathcal{X}$ such that $fu = hu$ and $fhu = hfu$; $gv = kv$ and $gkv = kgv$.

First step: we prove that $fu = gv$. Suppose that $\|fu - gv\| > 0$. Then, by using inequality (1.2) we get

$$\begin{aligned} \|fu - gv\|^p \le\ & \varphi(a\|hu - kv\|^p + (1-a)\max\{\alpha\|fu - hu\|^p, \\ & \beta\|gv - kv\|^p, \|fu - hu\|^{\frac{p}{2}}\|fu - kv\|^{\frac{p}{2}}, \\ & \|fu - kv\|^{\frac{p}{2}}\|gv - hu\|^{\frac{p}{2}}, \\ & \frac{1}{2}(\|fu - hu\|^p + \|gv - kv\|^p)\}); \end{aligned}$$

i.e.,

$$\begin{aligned} \|fu - gv\|^p &\le \varphi(a\|fu - gv\|^p + (1-a)\|fu - gv\|^p) \\ &= \varphi(\|fu - gv\|^p) \\ &< \|fu - gv\|^p \end{aligned}$$

which is a contradiction. Thus, we have $hu = fu = gv = kv$.

Second step: we claim that $ffu = fu = hfu$. If not, then, $\|f^2u - fu\| > 0$ and the use of inequality (1.2) gives

$$\begin{aligned} \|f^2u - fu\|^p &= \|ffu - gv\|^p \\ &\le \varphi(a\|hfu - kv\|^p + (1-a)\max\{\alpha\|ffu - hfu\|^p, \\ &\quad \beta\|gv - kv\|^p, \|ffu - hfu\|^{\frac{p}{2}}\|ffu - kv\|^{\frac{p}{2}}, \\ &\quad \|ffu - kv\|^{\frac{p}{2}}\|gv - hfu\|^{\frac{p}{2}}, \\ &\quad \frac{1}{2}(\|ffu - hfu\|^p + \|gv - kv\|^p)\}); \end{aligned}$$

that is,

$$\begin{aligned} \|ffu - fu\|^p &\le \varphi(\|ffu - fu\|^p) \\ &< \|ffu - fu\|^p \end{aligned}$$

this contradiction implies that $ffu = fu = hfu$. Similarly, we can prove that $ggv = gv = kgv$. Put $fu = hu = gv = kv = t$, we conclude that t is a common fixed point of maps f, g, h and k.

Third step: Suppose that there is another common fixed point of maps f, g, h and k called z, then, $\|t-z\|>0$. By inequality (1.2) we obtain

$$\begin{aligned}\|t-z\|^p &= \|ft-gz\|^p\\ &\le \varphi(a\|ht-kz\|^p+(1-a)\max\{\alpha\|ft-ht\|^p,\\ &\quad \beta\|gz-kz\|^p, \|ft-ht\|^{\frac{p}{2}}\|ft-kz\|^{\frac{p}{2}},\\ &\quad \|ft-kz\|^{\frac{p}{2}}\|gz-ht\|^{\frac{p}{2}},\\ &\quad \frac{1}{2}(\|ft-ht\|^p+\|gz-kz\|^p)\})\\ &= \varphi(\|t-z\|^p)\\ &< \|t-z\|^p.\end{aligned}$$

The above contradiction demands that $z=t$.

Corollary 1 *Let f, g, h, k be as in Theorem 2. Suppose that these maps satisfy instead of inequality (1.2) the next one*

$$\begin{aligned}\|fx-gy\|^p &\le \varphi(a\|hx-ky\|^p+(1-a)\max\{\alpha\|fx-hx\|^p,\\ &\quad \beta\|gy-ky\|^p, \|fx-hx\|^{\frac{1}{2}}\|fx-ky\|^{\frac{1}{2}},\\ &\quad \|fx-ky\|^{\frac{1}{2}}\|gy-hx\|^{\frac{1}{2}},\\ &\quad \frac{1}{2}(\|fx-hx\|+\|gy-ky\|)\}^p);\end{aligned}$$

for all $x,y\in\mathcal{X}$, where $\varphi, a, \alpha, \beta$ and p are as in Theorem 2, then, the four maps have a unique common fixed point.

Corollary 2 *If we replace inequality (1.2) in Theorem 2 with the following one*

$$\|fx-gy\|^p \le \varphi(a\|hx-ky\|^p+(1-a)\|fx-ky\|^{\frac{p}{2}}\|gy-hx\|^{\frac{p}{2}});$$

for all $x,y\in\mathcal{X}$, where φ, a and p are as in Theorem 2, then, f, g, h and k have a unique common fixed point.

We finish our work by giving the next result.

Theorem 3 *Let $\{f_i\}$, $i=1,2,\ldots$, h and k be self-maps of a normed space $(\mathcal{X},\|.\|)$ such that*
(i) pairs of maps $\{f_1,h\}$ and $\{f_n,k\}$, $n>1$ are occasionally weakly compatible,

(ii) the inequality

$$\|f_1x - f_ny\|^p \leq \varphi(a\|hx - ky\|^p + (1-a)\max\{\alpha\|f_1x - ky\|^p, \beta\|f_ny - hx\|^p\})$$

holds for all $x, y \in \mathcal{X}$, where $\alpha, \beta, \varphi, p$ are as in Theorem 2, $0 < a < 1$ provided that $a + (1-a)\max\{\alpha, \beta\} < 1$, then, all f_i, h and k have a unique common fixed point.

Proof. Since pairs $\{f_1, h\}$ and $\{f_n, k\}$, $n = 2, 3, \ldots$ are occasionally weakly compatible, then, as in proof of Theorem 2, there are two elements u and v in $\mathcal{X}$ such that $f_1u = hu$ and $f_1hu = hf_1u$; $f_nv = kv$ and $f_nkv = kf_nv$.
First, we prove that $f_1u = f_nv$. Indeed, let $f_1u \neq f_nv$, then, inequality (ii) gives

$$\begin{aligned}
\|f_1u - f_nv\|^p &\leq \varphi(a\|hu - kv\|^p + (1-a)\max\{\alpha\|f_1u - kv\|^p, \beta\|f_nv - hu\|^p\}) \\
&= \varphi([a + (1-a)\max\{\alpha, \beta\}]\|f_1u - f_nv\|^p) \\
&< [a + (1-a)\max\{\alpha, \beta\}]\|f_1u - f_nv\|^p \\
&< \|f_1u - f_nv\|^p
\end{aligned}$$

which is a contradiction. Hence, we have $f_1u = f_nv = hu = kv$.
Now, if $f_n^2v \neq f_nv$, then, by condition (ii) we have

$$\begin{aligned}
\|f_nv - f_n^2v\|^p &= \|f_1u - f_nf_nv\|^p \\
&\leq \varphi(a\|hu - kf_nv\|^p + (1-a)\max\{\alpha\|f_1u - kf_nv\|^p, \beta\|f_nf_nv - hu\|^p\}) \\
&= \varphi([a + (1-a)\max\{\alpha, \beta\}]\|f_nv - f_n^2v\|^p) \\
&< [a + (1-a)\max\{\alpha, \beta\}]\|f_nv - f_n^2v\|^p \\
&< \|f_nv - f_n^2v\|^p
\end{aligned}$$

a contradiction. Thus, $f_nf_nv = f_nv = kf_nv$. Similarly, $f_1f_1u = f_1u = hf_1u$. Put $hu = f_1u = f_nv = kv = t$, then, t is a common fixed point of maps $\{f_i\}_{i\geq 1}$, h and k.
The uniqueness of the common fixed point follows immediately from inequality (ii).

Remark 1 *In this paper, we proved a unique common fixed point of several maps in normed spaces by using the weaker condition of compatibility called occasionally weak compatibility due to Al-Thagafi and Shahzad without calling inclusions between images of maps. Hence, our result is more general than results in [3] and references therein and say that the weak compatibility is the least condition of maps to have common fixed points is not true.*

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