**General Mathematics Vol. 18, No. 3 (2010), 73–81**

# On the composite Bernstein type cubature formula [1]

Dan Bărbosu, Dan Miclăuş

**Abstract**

Considering a given function $f \in C([0,1] \times [0,1])$, the bivariate interval $[0,1] \times [0,1]$ is divided in $mn$ equally spaced bivariate subintervals $\left[\frac{k-1}{m}, \frac{k}{m}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]$, $k = \overline{1,m}$, $j = \overline{1,n}$. On each such type of subintervals the Bernstein bivariate approximation formula is applied and a corresponding Bernstein type cubature formula is obtained. Making the sum of mentioned cubature formulas, the composite Bernstein type cubature formula is obtained. The coefficients of above formula are determined and an upper-bound for its remainder term is given.

**2010 Mathematics Subject Classification:** 65D32, 41A10

**Key words and phrases:** Bernstein bivariate operator, Bernstein bivariate approximation formula, Bernstein quadrature formula, Bernstein cubature formula, Bivariate divided difference, Remainder term

## 1 Preliminaries

Let us to denote $\mathbb{N} = \{1, 2, ...\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
The Bernstein bivariate operator $B_{m,n} : C([0,1] \times [0,1]) \to C([0,1] \times [0,1])$ is defined for any $f \in C([0,1] \times [0,1])$, any $(x,y) \in [0,1] \times [0,1]$ and any $m, n \in \mathbb{N}$, by

$$(1) \qquad (B_{m,n}f)(x,y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) f\left(\frac{k}{m}, \frac{j}{n}\right),$$

[1] *Received 24 April, 2009*

*Accepted for publication (in revised form) 18 May, 2010*

where

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k} \tag{2}$$

and

$$p_{n,j}(y) = \binom{n}{j} y^j (1-y)^{n-j} \tag{3}$$

are the fundamental Bernstein's polynomials. The polynomial (1) is called the Bernstein bivariate polynomial.
Let $f \in C([0,1] \times [0,1])$ be given. The following equality

$$f(x,y) = (B_{m,n}f)(x,y) + (R_{m,n}f)(x,y) \tag{4}$$

is known as the Bernstein bivariate approximation formula, where $R_{m,n}$ is the remainder operator associated to the Bernstein bivariate operator $B_{m,n}$, i.e. $R_{m,n}f$ is the remainder term of the bivariate approximation formula (4). Regarding the remainder term of (4), D. Bărbosu and O. T. Pop [7] established the following:

**Theorem 1** *For any $f \in C([0,1] \times [0,1])$ and any $(x,y) \in [0,1] \times [0,1]$, the remainder term of (4) can be expressed under the form*

(5)

$$\begin{aligned}(R_{m,n}f)(x,y) = &- \frac{x(1-x)}{m} \sum_{k=0}^{m-1} \sum_{j=0}^{n} p_{m-1,k}(x) p_{n,j}(y) \begin{bmatrix} x, \frac{k}{m}, \frac{k+1}{m} & \\ \frac{j}{m} & ;f \end{bmatrix} \\ &- \frac{y(1-y)}{n} \sum_{k=0}^{m} \sum_{j=0}^{n-1} p_{m,k}(x) p_{n-1,j}(y) \begin{bmatrix} \frac{k}{m} & \\ y, \frac{j}{n}, \frac{j+1}{n} & ;f \end{bmatrix} \\ &+ \frac{xy(1-x)(1-y)}{mn} \sum_{k=0}^{m-1} \sum_{j=0}^{n-1} p_{m-1,k}(x) p_{n-1,j}(y) \begin{bmatrix} x, \frac{k}{m}, \frac{k+1}{m} & \\ y, \frac{j}{k}, \frac{j+1}{n} & ;f \end{bmatrix}.\end{aligned}$$

In (5) the brackets denote the bivariate divided differences.

**Theorem 2** ([7]) *Let be $p, q \in \mathbb{N}_0$, $a \le x_0 < x_1 < ... < x_p \le b$, $c \le y_0 < y_1 < ... < y_q \le d$ and the function $f : [a,b] \times [c,d] \to \mathbb{R}$ be given. If $f \in C^{(p-1,q-1)}([a,b] \times [c,d])$ and there exists $\frac{\partial^{p+q} f}{\partial x^p \partial y^q}$ on $]a,b[\times]c,d[$ then, there exists $(\xi, \eta) \in ]a,b[\times]c,d[$ such that*

$$\begin{bmatrix} x_0, x_1, \ldots, x_p & \\ y_0, y_1, \ldots, y_q & ;f \end{bmatrix} = \frac{1}{p!q!} \frac{\partial^{p+q} f}{\partial x^p \partial y^q}(\xi, \eta). \tag{6}$$

**Theorem 3** ([7]) *Let* $f : [0,1] \times [0,1] \to \mathbb{R}$ *be a function with the properties that* $f \in C([0,1] \times [0,1])$*, there exists* $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ *on* $]0,1[\times]0,1[$ *and* $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ *are bounded on* $]0,1[\times]0,1[$*. Then, the inequalities*

(7)
$$\begin{aligned} |(R_{m,n}f)(x,y)| &\le \frac{x(1-x)}{2m}M_1[f] + \frac{y(1-y)}{2n}M_2[f] + \frac{xy(1-x)(1-y)}{4mn}M_3[f] \\ &\le \frac{1}{8m}M_1[f] + \frac{1}{8n}M_2[f] + \frac{1}{64mn}M_3[f] \end{aligned}$$

*hold, for any* $(x,y) \in [0,1] \times [0,1]$ *and any* $m, n \in \mathbb{N}$*, where*

$$\begin{aligned} M_1[f] &= \sup_{(x,y)\in]0,1[\times]0,1[} \left| \frac{\partial^2 f}{\partial x^2}(x,y) \right|, \\ M_2[f] &= \sup_{(x,y)\in]0,1[\times]0,1[} \left| \frac{\partial^2 f}{\partial y^2}(x,y) \right|, \\ M_3[f] &= \sup_{(x,y)\in]0,1[\times]0,1[} \left| \frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \right|. \end{aligned} \tag{8}$$

Integrating the Bernstein bivariate approximation formula (4) one arrives to the following Bernstein's cubature formula [6]:

$$\int_0^1 \int_0^1 f(x,y)dxdy = \sum_{i=0}^{m} \sum_{j=0}^{n} A_{i,j} f\left(\frac{i}{m}, \frac{j}{n}\right) + R_{m,n}[f], \tag{9}$$

where

$$A_{i,j} = \frac{1}{(m+1)(n+1)}, \quad i = \overline{0,m}, \quad j = \overline{0,n} \tag{10}$$

and

$$|R_{m,n}[f]| \le \frac{1}{12m}M_1[f] + \frac{1}{12n}M_2[f] + \frac{1}{144mn}M_3[f], \tag{11}$$

where $M_1[f]$, $M_2[f]$ and $M_3[f]$ were defined at (8).

The focus of the present paper is to construct the composite Bernstein type cubature formula. For this aim, the bivariate interval $[0,1] \times [0,1]$ will be divided in $mn$ equally spaced bivariate subintervals $\left[\frac{k-1}{m}, \frac{k}{m}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]$, $k = \overline{1,m}$, $j = \overline{1,n}$. On each such type of subintervals $\left[\frac{k-1}{m}, \frac{k}{m}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]$, the Bernstein's cubature formula (9) will be applied. Next, adding the mentioned cubature formulas, the composite Bernstein cubature formula on $[0,1] \times [0,1]$ will be obtained.

## 2 Main results

We start by the simple results contained in the following two lemmas.

**Lemma 1** *Suppose that* $a, b, c, d \in \mathbb{R}$, $a < b$ *and* $c < d$, $f \in C([a,b] \times [c,d])$. *Then, the bivariate Bernstein polynomial associated to the function* $f$ *is defined by*

$$(12)\quad (B_{m,n}f)(x,y) = \frac{1}{(b-a)^m(d-c)^n}\sum_{k=0}^{m}\sum_{j=0}^{n}\binom{m}{k}\binom{n}{j}(x-a)^k(b-x)^{m-k} \cdot (y-c)^j(d-y)^{n-j} f\left(a + k\frac{b-a}{m}, c + j\frac{d-c}{n}\right).$$

**Proof.** The parametric extensions [5] of the Bernstein's univariate operator [8] are defined by

$$(13)\quad (B_m^x f)(x,y) = \frac{1}{(b-a)^m(d-c)^n}\sum_{k=0}^{m}\sum_{j=0}^{n}\binom{m}{k}\binom{n}{j}(x-a)^k(b-x)^{m-k} \cdot (y-c)^j(d-y)^{n-j} f\left(a + k\frac{b-a}{m}, y\right),$$

respectively

$$(14)\quad (B_n^y f)(x,y) = \frac{1}{(b-a)^m(d-c)^n}\sum_{k=0}^{m}\sum_{j=0}^{n}\binom{m}{k}\binom{n}{j}(x-a)^k(b-x)^{m-k} \cdot (y-c)^j(d-y)^{n-j} f\left(x, c + j\frac{d-c}{n}\right).$$

Their tensorial product [9] is the bivariate Bernstein operator (12).

**Lemma 2** *Suppose that* $a, b, c, d \in \mathbb{R}$, $a < b$, $c < d$ *and let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be a function with the properties that* $f \in C^{2,2}([a,b] \times [c,d])$, *there exists* $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ *on* $]a,b[\times]c,d[$ *and* $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ *are bounded on* $]a,b[\times]c,d[$. *Then, the remainder term of the Bernstein bivariate approximation formula on* $[a,b] \times [c,d]$ *verifies the following inequality*

$$(15)\quad |(R_{m,n}f)(x,y)| \le \frac{(x-a)(b-x)}{2m(b-a)^2}M_1'[f] + \frac{(y-c)(d-y)}{2n(d-c)^2}M_2'[f] + \frac{(x-a)(b-x)(y-c)(d-y)}{4mn(b-a)^2(d-c)^2}M_3'[f],$$

*for any* $(x,y) \in ]a,b[\times]c,d[$ *and any* $m,n\in\mathbb{N}$*, where*

$$
\begin{aligned}
M_1'[f] &= \sup_{(x,y)\in]a,b[\times]c,d[} \left|\frac{\partial^2 f}{\partial x^2}(x,y)\right|,\\
M_2'[f] &= \sup_{(x,y)\in]a,b[\times]c,d[} \left|\frac{\partial^2 f}{\partial y^2}(x,y)\right|,\\
M_3'[f] &= \sup_{(x,y)\in]a,b[\times]c,d[} \left|\frac{\partial^4 f}{\partial x^2\partial y^2}(x,y)\right|.
\end{aligned}
\tag{16}
$$

**Proof.** One applies (7) and the method of parametric extensions [9].

In what follows, let us to consider the bivariate interval $[0,1]\times[0,1]$ divided in the $mn$ equally spaced bivariate subintervals $\left[\frac{k-1}{m},\frac{k}{m}\right]\times\left[\frac{j-1}{n},\frac{j}{n}\right]$, $k=\overline{1,m}$, $j=\overline{1,n}$. In each interval $\left[\frac{k-1}{m},\frac{k}{m}\right]\times\left[\frac{j-1}{n},\frac{j}{n}\right]$ one considers the distinct knots $x_h=\frac{kp-p+h}{mp}$, $h=\overline{0,p}$, respectively $y_l=\frac{jq-q+l}{nq}$, $l=\overline{0,q}$.
Applying Lemma 1 yields the following Bernstein type bivariate polynomial

(17)

$$
\begin{aligned}
(B_{p,k,q,j}f)(x,y) = m^p n^q \sum_{h=0}^{p}\sum_{l=0}^{q}\binom{p}{h}\binom{q}{l}\left(x-\frac{k-1}{m}\right)^h\left(\frac{k}{m}-x\right)^{p-h}\\
\cdot\left(y-\frac{j-1}{n}\right)^l\left(\frac{j}{n}-y\right)^{q-l} f\left(\frac{kp-p+h}{mp},\frac{jq-q+l}{nq}\right).
\end{aligned}
$$

The following Bernstein type bivariate approximation formula

$$
f(x,y) = (B_{p,k,q,j}f)(x,y) + (R_{p,k,q,j}f)(x,y) \tag{18}
$$

holds, on any interval $\left[\frac{k-1}{m},\frac{k}{m}\right]\times\left[\frac{j-1}{n},\frac{j}{n}\right]$, $k=\overline{1,m}$, $j=\overline{1,n}$.
For any $f\in C^{2,2}([0,1]\times[0,1])$ the following upper-bound estimation for the bivariate remainder term

(19)

$$
\begin{aligned}
|(R_{p,k,q,j}f)(x,y)| \le \frac{m\left(x-\frac{k-1}{m}\right)\left(\frac{k}{m}-x\right)}{2}M_1''[f] + \frac{n\left(y-\frac{j-1}{n}\right)\left(\frac{j}{n}-y\right)}{2}M_2''[f]\\
+\frac{mn\left(x-\frac{k-1}{m}\right)\left(\frac{k}{m}-x\right)\left(y-\frac{j-1}{n}\right)\left(\frac{j}{n}-y\right)}{4}M_3''[f]
\end{aligned}
$$

holds, where

$$M_1''[f] = \sup_{(x,y)\in\left]\frac{k-1}{m},\frac{k}{m}\right[\times\left]\frac{j-1}{n},\frac{j}{n}\right[} \left|\frac{\partial^2 f}{\partial x^2}(x,y)\right|,$$

(20) $$M_2''[f] = \sup_{(x,y)\in\left]\frac{k-1}{m},\frac{k}{m}\right[\times\left]\frac{j-1}{n},\frac{j}{n}\right[} \left|\frac{\partial^2 f}{\partial y^2}(x,y)\right|,$$

$$M_3''[f] = \sup_{(x,y)\in\left]\frac{k-1}{m},\frac{k}{m}\right[\times\left]\frac{j-1}{n},\frac{j}{n}\right[} \left|\frac{\partial^4 f}{\partial x^2\partial y^2}(x,y)\right|.$$

**Theorem 4** *If $f \in C([0,1]\times[0,1])$, the coefficients of the Bernstein type cubature formula*

(21)
$$\int_{\frac{k-1}{m}}^{\frac{k}{m}}\int_{\frac{j-1}{n}}^{\frac{j}{n}} f(x,y)dxdy = \sum_{h=0}^{p}\sum_{l=0}^{q} A_{h,k,l,j} f\left(\frac{kp-p+h}{mp}, \frac{jq-q+l}{nq}\right) + R_{k,j}[f]$$

*can be expressed under the form*

(22) $$A_{h,k,l,j} = \frac{1}{mn(p+1)(q+1)}, \quad h=\overline{0,p}, \quad l=\overline{0,q}.$$

**Proof.** Integrating (18) on $\left[\frac{k-1}{m},\frac{k}{m}\right]\times\left[\frac{j-1}{n},\frac{j}{n}\right]$, $k=\overline{1,m}$, $j=\overline{1,n}$ and taking (17) into account, yields

$$A_{h,k,l,j} = m^p n^q \binom{p}{h}\binom{q}{l} \int_{\frac{k-1}{m}}^{\frac{k}{m}}\int_{\frac{j-1}{n}}^{\frac{j}{n}} \left(x-\frac{k-1}{m}\right)^h \left(\frac{k}{m}-x\right)^{p-h}$$
$$\cdot \left(y-\frac{j-1}{n}\right)^l \left(\frac{j}{n}-y\right)^{q-l} dxdy$$
$$= m^p n^q \binom{p}{h}\binom{q}{l} \int_{\frac{k-1}{m}}^{\frac{k}{m}} \left(x-\frac{k-1}{m}\right)^h \left(\frac{k}{m}-x\right)^{p-h} dx$$
$$\cdot \int_{\frac{j-1}{n}}^{\frac{j}{n}} \left(y-\frac{j-1}{n}\right)^l \left(\frac{j}{n}-y\right)^{q-l} dy$$

$$= m^p n^q \binom{p}{h}\binom{q}{l} \int_0^1 \frac{1}{m^{p+1}} t^h (1-t)^{p-h} dt \int_0^1 \frac{1}{n^{q+1}} u^l (1-u)^{q-l} du$$

$$= \frac{1}{mn} \binom{p}{h}\binom{q}{l} \int_0^1 t^h (1-t)^{p-h} dt \int_0^1 u^l (1-u)^{q-l} du.$$

The integral $\int_0^1 x^i (1-x)^{n-i} dx$ is the Euler function of first kind $B(i+1, n-i+1)$. Taking the well known properties of Euler function of first kind into account, it follows that

$$\begin{aligned} A_{h,k,l,j} &= \frac{1}{mn} \binom{p}{h}\binom{q}{l} B(h+1, p-h+1) B(l+1, q-l+1) \\ &= \frac{1}{mn} \frac{p!}{h!(p-h)!} \frac{h!(p-h)!}{(p+1)!} \frac{q!}{l!(q-l)!} \frac{l!(q-l)!}{(q+1)!} \\ &= \frac{1}{mn(p+1)(q+1)}. \end{aligned}$$

**Theorem 5** *Let* $f \in C^{2,2}([0,1]\times[0,1])$ *be given so that there exists* $\frac{\partial^2 f}{\partial x^2 \partial y^2}$ *on* $]0,1[\times]0,1[$ *and* $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ *are bounded on* $]0,1[\times]0,1[$. *Then the following upper-bound estimation for the bivariate remainder term of the Bernstein type cubature formula* (21) *is*

$$|R_{k,j}[f]| \le \frac{M_1''[f]}{12m^2 n} + \frac{M_2''[f]}{12mn^2} + \frac{M_3''[f]}{144m^2 n^2}, \tag{23}$$

*where* $M_1''[f]$, $M_2''[f]$, $M_3''[f]$ *are given at* (20).

**Proof.** The inequality (23) follows by integrating the Bernstein type bivariate approximation formula (18) and taking the inequality (19) into account.

**Theorem 6** *For any* $f \in C^{2,2}([0,1]\times[0,1])$, *the following composite Bernstein type cubature formula*

$$\int_0^1\int_0^1 f(x,y)dxdy = \frac{1}{mn(p+1)(q+1)} \sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{h=0}^{p}\sum_{l=0}^{q} f\left(\frac{kp-p+h}{mp}, \frac{jq-q+l}{nq}\right) + R_{m,n}[f]. \tag{24}$$

*holds, where* $|R_{m,n}[f]|$ *were defined at* (11).

**Proof.** Adding the relation (21) for any $k = \overline{1,m}$, $j = \overline{1,n}$, we get the following composite Bernstein type cubature formula (24).

**Remark 1** *It is easily to see that we get the same result for the bivariate remainder term of the composite Bernstein type cubature formula, as the result obtained by D. Bărbosu and O. T. Pop in [6].*

**Corollary 1** *The following equality*

$$(25)\qquad \lim_{m,n\to\infty} \frac{1}{mn(p+1)(q+1)} \sum_{k=1}^{m}\sum_{j=1}^{n}\sum_{h=0}^{p}\sum_{l=0}^{q} f\left(\frac{kp-p+h}{mp}, \frac{jq-q+l}{nq}\right) = \int_0^1\int_0^1 f(x,y)dxdy$$

*holds.*

**Proof.** Yields immediately from Theorem 6, because $\lim\limits_{m,n\to\infty} |R_{m,n}[f]| = 0$.

## References

[1] O. Agratini, *Approximation by linear operators (Romanian)*, Presa Universitară Clujeană 2000

[2] D. Bărbosu, *The approximation of multivariate functions by boolean sums of linear operators of interpolatory type (Romanian)*, Ed. Risoprint, Cluj Napoca 2002

[3] D. Bărbosu, *Polynomial Approximation by Means of Schurer-Stancu type operators*, Ed. Universităţii de Nord, Baia Mare 2006

[4] D. Bărbosu, *On the Schurer-Stancu approximation formula*, Carpathian J. Math. **21**, 2005, 7-12

[5] D. Bărbosu, O. T. Pop, *A note on the GBS Bernstein's approximation formula*, Annals of the University of Craiova, Math. Comp. Sci. Ser. **35**, 2008, 1-6

[6] D. Bărbosu, O. T. Pop, *A note on the Bernstein's cubature formula*, General Mathematics **17** (3), 2009, 161-172

[7] D. Bărbosu, O. T. Pop, *On the Bernstein bivariate approximation formula*, Carphatian J. Math. **24** (3), 2008, 293-298

[8] D. Bărbosu, D. Miclăuş, *On the composite Bernstein type quadrature formula*, (to appear in Rev. Anal. Num. Théor. Approx.)

[9] F. J. Delvos, W. Schempp, *Boolean methods in interpolation and approximation*, Longmann Scientific and Technical 1989

[10] M. Ivan, *Elements of Interpolation Theory*, Mediamira Science Publisher, Cluj-Napoca 2004

[11] D. D. Stancu, *Quadrature formulas constructed by using certain linear positive operators*, Numerical Integration (Proc. Conf. Math. Res. Inst. Oberwolffach) (Basel) (G. Hammerlin, ed.), Birkhäuser 1982, 241-251

[12] D. D. Stancu, Gh. Coman, P. Blaga, *Numerical Analysis and Approximation Theory (Romanian)*, **II**, Presa Universitară Clujeană 2002

**Dan Bărbosu, Dan Miclăuş**
North University of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 430122 Baia Mare, Romania
e-mail: barbosudan@yahoo.com, danmiclausrz@yahoo.com