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On the exponential diophantine equations of the form $a^x - b^y \cdot c^z = \pm 1$

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Abstract

In this short paper we obtain by elementary methods some general results for some exponential diophantine equations of the form $a^x - b^y \cdot c^z = \pm 1$

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1 Introduction

This type of exponential diophantine equations was studied by D.Acu, L.J.Alex, L.S.Chen, J.H.Teng, Y.B.Wang.

Theorem 1 *[3] If p_1, p_2 are distinct prime numbers and not equal with 3, then the equation $7^x - p_1^y \cdot p_2^z = 1$ has no nonnegative integer solutions.*

Theorem 2 *[7] The equation $a^x - p_1^{y_1} p_2^{y_2} \dots p_k^{y_k} = \pm 1$ where a is a positive integer with $a > 1$ and $p_1, p_2, ..., p_k$ are distinct primes with $gcd(a, p_1 p_2 ... p_k) = 1$, has only finitely many positive integer solutions.*

Theorem 3 *[1] If a, b, c are distinct prime numbers and not equal with 2, then the equation $a^x - b^y \cdot c^z = \pm 1$ has no nonnegative integer solutions.*

2 New results

Theorem 4 *If a is a natural numbers, $a \equiv 1$ (mod p), and b, c are natural numbers with $bc \not\equiv 0$ (mod p), where p is a natural number with $p \geq 2$, then the equations $a^x - b^y \cdot c^z = 1$ have no nonnegative integer solutions.*

Proof. The equation $a^x - b^y \cdot c^z = 1$ is equivalent with

(1) $$a^x - 1 = b^y \cdot c^z$$

From $bc \not\equiv 0$ (mod p) we obtain

(2) $$b^y \cdot c^z \not\equiv 0 \ (mod\ p)$$

Because $a \equiv 1$ (mod p) we obtain $a^x \equiv 1$ (mod p), and thus we conclude

(3) $$a^x - 1 \equiv 0 \ (mod\ p)$$

Using (1), (2), (3) we find a contradiction. We conclude that the equations $a^x - b^y \cdot c^z = 1$ have no nonnegative integer solutions.

Remark 1 *From theorem 4 with $a = 7, p = 3, b = p_1, c = p_2$, where p_1, p_2 are distinct prime numbers and not equal with 3, we obtain the result from theorem 1.*

Remark 2 *From theorem 4 with a, b, c distinct prime numbers and not equal with 2, and $p = 2$, we obtain a half of results from theorem 3.*

Theorem 5 *If a is a natural number, $a \equiv -1$ (mod p), and b, c are natural numbers with $bc \equiv 0$ (mod p), where p is a natural number with $p \geq 2$, then the equations $a^x - b^y \cdot c^z = -1$ have no nonnegative integer solutions with the form $(2k + 1, y, z)$.*

Proof. By using congruences with p modulus, in a similar way with the proof of the theorem 4, we obtain the assertion.

Theorem 6 *If a is a natural number, $a \equiv 1$ (mod p), and $p_1, p_2, ..., p_k$ are natural numbers with $\prod_{i=1}^{k} p_i \not\equiv 0$ (mod p), where p is a natural number with $p \geq 2$, then the equations $a^x - \prod_{i=1}^{k} p_i^{y_i} = 1$ have no nonnegative integer solutions*

Proof. The equation $a^x - \prod_{i=1}^{k} p_i^{y_i} = 1$ is equivalent with $a^x - 1 = \prod_{i=1}^{k} p_i^{y_i}$. Using congruences with p modulus we show that $a^k - 1 \equiv 0 \pmod{p}$ and $\prod_{i=1}^{k} p_i^{y_i} \not\equiv 0 \pmod{p}$. Thus we find a contradiction and we obtain the assertion.

Remark 3 *It is easy to see that the Theorem 6 is complementary to the Theorem 2.*

Theorem 7 *The only solutions of the equation $3^x - 5^y \cdot 2^z = -1$ nonnegative integers are (0, 0, 1), (1, 0, 2) and (2, 1, 1).*

Proof. The equation $3^x - 5^y \cdot 2^z = -1$ is equivalent with

$$3^x + 1 = 5^y \cdot 2^z \tag{4}$$

If $z \geq 3$ then $2^z \equiv 0 \pmod{8}$. Using (4) we obtain $3^x + 1 \equiv 0 \pmod{8}$.

By using congruences with 8 modulus for $x = 0,\ x = 2p$ and respectively $x = 2p + 1$, we conclude that $3^x + 1 \not\equiv 0 \pmod{8}$.

If $y \geq 2$ then $5^y \equiv 0 \pmod{25}$. Using (4) we obtain $3^x + 1 \equiv 0 \pmod{25}$. By using congruences with 25 modulus $x = 4q,\ x = 4q + 1,\ x = 4q + 2$, and respectively $x = 4q + 3$, we conclude that $3^x + 1 \not\equiv 0 \pmod{25}$.

Now we conclude that

$$y \in \{0, 1\} \text{ and } z \in \{0, 1, 2\} \tag{5}$$

By a simple calculation we obtain from (4) and (5) the only solutions (0, 0, 1), (1, 0, 2) and (2, 1, 1).

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