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An application of the Glaeser theorem to PDE's

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Abstract

We consider a system of partial differential equations of the first order such that the conditions on derivatives imply that solutions are symmetric or even functions. We apply Glaeser's generalization of the Newton theorem to find all C^∞ solutions of such system. These equations may be solved using the classical methods, however, we would like to show how the important theorem of Glaeser has applications in solving PDE's.

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1 Preliminaries

We denote by $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ for $k = 1, 2, ..., n$ the *elementary symmetric functions*

$$\sigma_k(x_1, ..., x_n) = \sum_{1 \le i_1 < ... < i_k \le n} x_{i_1} ... x_{i_k}.$$

We also put $\theta := (\sigma_1, ..., \sigma_n)$. Then it is known that θ has its Jacobian different from zero on an open dense subset of $\mathbb{R}^n$ (cf. [4]). There is the classical theorem of Newton which says that any symmetric polynomial in n variables is a composition of θ with other polynomial. It was until 1963 that this theorem was generalized by Glaeser for C^∞ symmetric functions (cf. [4]). This theorem is generally not true for C^k symmetric functions. Barbançon considered this case and obtained a factorisation with a decrease of the class of differentiability (cf. [1]).

In all of this paper we supose that U is an open connected subset of $\mathbb{R}^n$. Moreover we suppose that U is invariant with respect to the permutations of the coordinates. We shall look for the solutions of our partial differential equations on the set U.

We consider a system of partial differential equations

$$u_{x_i} = u_{x_j} \tag{1}$$

where $i, j = 1, 2, ..., n$ and $u : U \to \mathbb{R}$.

Remark 1 *These equations may be solved using the classical theory of characteristic (cf. [3]). Another way is to apply the Frobenius theorem.*

We shall sketch the latter one. Let M be a differentiable manifold; means here at least of the class C^1. Suppose that $\mathcal{A}$ is a set of the differentiable vector fields on M such that the function $x \to dim\{X_x : X \in \mathcal{A}\}$ is constant on M. Then one of the versions of Frobenius theorem says that $\mathcal{A}$ is completely integrable if $\mathcal{A}$ is involutive (cf. [6]). The complete integrability of $\mathcal{A}$ means that there exists a foliation on M such that the tangent spaces to its leaves are equal to $span\{X_x : X \in \mathcal{A}\}$ as x varies in M. The leaves of the foliation may be obtained from the orbits of the flows of the vector fields from $\mathcal{A}$. As consequences of the Frobenius theorem we get that if $\mathcal{A}$ is involutive then the solutions of the system

$$\partial_X u = 0, \quad X \in \mathcal{A}. \tag{2}$$

are exactly the functions which are constant on the leaves of the foliation determined by $\mathcal{A}$. In the case of equations (1) the set $\mathcal{A}$ consists of the vector fields

$$\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \tag{3}$$

for $i, j = 1, ..., n$. Then the leaves of the foliation determined by (3) are given by the equation $x_1 + ... + x_n = const$. Hence it follows that the solutions of (1) are given <u>locally</u> by $u(x_1, ..., x_n) = \varphi(x_1, ..., x_n)$ where φ is differentiable on an open subset of $\mathbb{R}$.

However, we shall apply here the deep theorem of Glaeser to resolve equations (1). The idea is to prove that if u satisfies (1) then u has to be symmetric; then we apply the theorem of Glaeser to get the solutions.

2 Symmetric solutions

Lemma 1 *If $u : U \to \mathbb{R}$ is a C^1 solution of (1) then u is a symmetric function.*

Proof. We fix $x_3, ..., x_n$ and define

$$U_1 := \{(x, y) \in \mathbb{R}^2 | (x, y, x_3, ..., x_n) \in U\}.$$

Consider $\phi : U_1 \to \mathbb{R}$ defined as follows:

$$\phi(x, y) := u(x, y, x_3, ..., x_n) - u(y, x, x_3, ..., x_n).$$

Then we have $\phi(x, x) = 0$ for all $(x, x) \in U_1$. We consider the auxiliary function
$\mu_1(t) = \phi(t, a - t)$ where $a \in \mathbb{R}$. Then we have that

$$\mu_1'(t) = \phi_x(t, a - t) - \phi_y(t, a - t)$$

$$= u_{x1}(t, a - t, x_3, ..., x_n) - u_{x2}(t, a - t, x_3, ..., x_n) = 0.$$

Hence μ_1 is constant. Then it follows that ϕ is zero on the line $x - y = 0$ and is constant on the lines $x + y = a$. Hence it follows that ϕ is constant equal zero. It means that u is symmetric with respect to x_1 and x_2 while $x_3, ..., x_n$ are fixed. Analogically one may show that u is symmetric with respect to all of the variables.

Theorem 1 *The function u is a C^∞ solution of (1) if there exists a C^∞ map $\varphi : \mathbb{R} \to \mathbb{R}$ such that $u = \varphi \circ \sigma_1$.*

Proof. Suppose that u is a C^∞ solution of (1). From Lemma (1) it follows that u is symmetric. Then from the theorem of Glaeser (cf.[4]) it follows that there exists a C^∞ map $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $u = \varphi \circ \theta$. The system (1) is equivalent to

$$u_{x1} = u_{xi}$$

where $i = 2, ..., n$. Hence we get that

$$\sum_{k=1}^{n} \frac{\partial \varphi}{\partial y_k} \cdot \frac{\partial \sigma_k}{\partial x_1} = \sum_{k=1}^{n} \frac{\partial \varphi}{\partial y_k} \cdot \frac{\partial \sigma_k}{\partial x_i}$$

where $i = 2, ..., n$ and then

$$\sum_{k=1}^{n} \frac{\partial \varphi}{\partial y_k} \left(\frac{\partial \sigma_k}{\partial x_1} - \frac{\partial \sigma_k}{\partial x_i} \right) = 0. \tag{4}$$

It follows that $\frac{\partial \varphi}{\partial y_2}, ..., \frac{\partial \varphi}{\partial y_n}$ are solutions of the system of linear equations with the functional coefficients. The coefficients in (1) form $a(n-1)(n-1)$ matrix with the components

$$\left(\frac{\partial \sigma_k}{\partial x_1} - \frac{\partial \sigma_k}{\partial x_i} \right) \quad i, k = 2, ..., n. \tag{5}$$

Since

$$\frac{\partial \sigma_1}{\partial x_i} = 1$$

then for all $i = 1, ..., n$ the determinant of (1) is equal to

$$det \begin{pmatrix} \frac{\partial \sigma_1}{\partial x_1} & \frac{\partial \sigma_1}{\partial x_2} & \cdots & \frac{\partial \sigma_1}{\partial x_n} \\ \frac{\partial \sigma_2}{\partial x_1} & \frac{\partial \sigma_2}{\partial x_2} & \cdots & \frac{\partial \sigma_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \sigma_n}{\partial x_1} & \frac{\partial \sigma_n}{\partial x_2} & \cdots & \frac{\partial \sigma_n}{\partial x_n} \end{pmatrix} \tag{6}$$

Then the determinant in (5) is just the jacobian of the map θ. It is well known that this Jacobian is different from zero on an open and dense subset of $\mathbb{R}^n$ (cf.[4]) and then on an open and dense subset of U. Then it follows that

$$\frac{\partial \varphi}{\partial y_2} = \frac{\partial \varphi}{\partial y_3} = ... = \frac{\partial \varphi}{\partial y_n} = 0$$

as they are the unique solutions of (4) on the set U. Hence φ does not depend on the variables $y_2, ..., y_n$. It follows that if u is a solution of (1) then it has to be of the form $u = \varphi \circ \sigma_1$. On the other hand, one veryfies easily that each function which is a composition of σ_1 and φ ia a solution of (1). This is the end of proof.

3 Even solutions

Suppose now that A_i are the reflections in $\mathbb{R}^n$ defined by
$A_i(e_j) := (-1)^{\delta_{ij}}$ where $i, j = 1, ..., n$ and $e_1, ..., e_n$ is the canonical basis of $\mathbb{R}^n$.

Suppose that V is an open ball in $\mathbb{R}^n$ centered at the origin of the coordinates. Then V is invariant with respect to all of the reflections A_i where $i \in \{1, ..., n\}$.

We shall say that $u : V \to \mathbb{R}$ is even if for each $i \in \{1, ..., n\}$

$$u \circ A_i = u.$$

Let $Q : \mathbb{R}^n \to \mathbb{R}^n$ be the following function

$$Q(x_1, ..., x_n) := (x_1^2, ..., x_n^2).$$

We shall use the map Q to factorize functions which are even.

Lemma 2 *If $u : V \to \mathbb{R}$ is an even function of the class C^∞ then there exists a C^∞ function ψ on $\mathbb{R}^n$ such that*

$$u = \psi \circ Q. \tag{7}$$

Proof. Let us notice that the function Q satisfies conditions Θ_1, Θ_2, Θ_3, Θ_4 of Theorem II ([4]). We choose polynomials w_k on $\mathbb{R}^n$ which tend to φ in the topology of uniform convergence together with an arbitrary order derivatives on compact subsets of $\mathbb{R}^n$. We denote by $\mathcal{F}(V)$ the set of real valued functions on V. Then we define an operator $P : \mathcal{F}(V) \to \mathcal{F}(V)$ which makes functions even and which is the identity on even functions; namely if $h : V \to \mathbb{R}$ then

$$P(h)(x_1, ..., x_n) = \frac{1}{2^n} \sum_{p \subset 1,...,n} h(\epsilon_1(p)x_1, ..., \epsilon_n(p)x_n)$$

where

$$\epsilon_i(p) = \begin{cases} 1, & \text{if } i \in p \\ -1, & \text{otherwise} \end{cases}$$

Then we obtain a sequence of even polynomials $P(w_l)$ which converges to u. We observe that this lemma is valid for functions which are polynomials in $x_1, ..., x_n$; this may be shown in a simple way as is done for one variable in [7]. Then for each natural l there exists a C^∞ map ψ_l such that $P(w_l) = \psi_l \circ Q$. In fact, ψ_l may be shown to be also polynomials. By the theorem of Glaeser (cf. [4]) the set of the functions

$$\mathcal{A}_Q(V) := \{h \circ Q|_V \ \ | \ \ h \in C^\infty(\mathbb{R}^n)\}$$

is closed in the set of all C^∞ functions on V. Hence $u \in \mathcal{A}_Q(V)$; in the other words it means that there exists a real valued C^∞ function ψ on $\mathbb{R}^n$ such that $u = \psi \circ Q$. This ends the proof.

Remark 2 *It is clear that any functions, of an arbitrary class, which has a decomposition as in (2) is even. We also would like to underline that in [5] we use the above method to prove a similar property.*

Theorem 2 *A function u of the class C^∞ on V is a solution of the system of partial differential equations*

$$x_j u_{x_i} = x_i u_{x_j} \tag{8}$$

if there exists a C^∞ map $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$u = \varphi \circ \sigma_1 \circ Q.$$

Proof. Let u be a solution of (2) and let $(x_1, x_2, ..., x_n) \in V$ then there exists $r, \alpha \in \mathbb{R}$ such that $x_1 = r\cos\alpha$ and $x_2 = r\sin\alpha$. Then we consider a map
$\gamma(t) = u(r\cos t, r\sin t, x_3, ..., x_n)$. After differentiating γ and applying (2) we get that $\gamma' = 0$. It follows that $\gamma(\alpha) = \gamma(\pi - \alpha)$. Hence $u(x_1, x_2, ..., x_n) =$ $= u(-x_1, x_2, ..., x_n)$ which means that u is even with respect to the first variable. Analogically, one proves that u is even with respect to all of the variables. Then there exists a C^∞ map $\psi : \mathbb{R} \to \mathbb{R}$ such that $u = \psi \circ Q$. Since u satisfies equations (2) then for each $i, j = 1, ..., n$ and for each $(x_1, ..., x_n) \in U$

$$x_i[2x_j\psi_{y_j}(x_1^2, ..., x_n^2)] = x_j[2x_i\psi_{y_i}(x_1^2, ..., x_n^2)].$$

Then it follows that ψ satisfies the following equations

$$\psi_{y_i} = \psi_{y_j}$$

for $i, j = 1, ..., n$. Now from (2) we have that there exists a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\psi = \varphi \circ \sigma_1$. Hence we get that $u = \varphi \circ \sigma_1 \circ Q$. On the other hand functions which are compositions of Q, σ_1 and any C^∞ function φ are clearly solutions of (2).

Remark 3 *One may solve equations (1) and (2) using, for example, the classical methods of characteristic and gets much better results because can find similar solutions with the lower class of the differentiability of φ. Unfortunately, the theorem of Newton is not valid for C^k functions. In general a function φ, if it exists, is of the lower class than k (cf. [1]). There are many generalizations of the Glaeser theorem (cf. [2] and the references there), which may find their applications in the other branches of mathematics.*

Remark 4 *The functions φ, ψ which exist and determine solutions of the systems (1) and (2) are unique when restricted to the image of σ_1.*

Remark 5 *Equations (2) may be solved using the Frobenius theorem again (cf. 1). We observe that the vector fields*

$$x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \qquad i, j = 1, ..., n \tag{9}$$

form an involutive family. The foliation obtained from (2) consists of submanifolds defined by $x_1^2 + ... + x_n^2 = const.$ Hence the local version of Theorem (2) follows.

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