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A preserving property of a Bernardi type operator

Mugur Acu

Abstract

The purpose of this note is to show a preserving property of a Bernardi type operator on a subclass of normalized analytic functions in open disc U.

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1 Introduction

Let $\mathcal{H}(\mathcal{U})$ be the set of functions wich are regular in the unit disc $\mathcal{U}$ and $A = \{f \in \mathcal{H}(\mathcal{U}) : f(0) = 0, f'(0) = 1\}$.

Let consider the integral operator $L_a : A \to A$ defined as:

$$(1) \qquad f(z) = L_a F(z) = \frac{1+a}{Z^a} \int_0^z F(t) \cdot t^{a-1} dt, \quad a \in \mathbb{C},\ Re\ a \geq 0.$$

In the case $a = 1, 2, 3, ...$ this operator was introduced by S.D. Bernardi and it was studied by many authors in different general uses. In the form (1) was used first time by N.N. Pascu.

2 Preliminary results

Let consider α and β real numbers with $0 \leq \beta < 1$ and $\alpha > 1$.

Definition 2.1 *Let $M(\alpha, \beta)$ be the subclass of A consisting of functions $f(z)$ which satisfy the inequality:*

$$\beta < Re\left\{\frac{zf'(z)}{f(z)}\right\}, \alpha, z \in \mathcal{U}. \tag{2}$$

Remark 2.1 *The class $M\alpha, \beta$ is introduced in relation with the classes $M(\alpha), M_n^*(\alpha)$ introduced and investigated in [4], [5] and [6].*

Remark 2.2 *If we consider h convex in unit $\mathcal{U}$, with $h(0) = 1$, which maps the unit disc into the convex domain included in right half plain $\Delta_{\alpha,\beta} = \{z \in \mathbb{C} : \beta < Rez < \alpha\}$, then from (2) we obtain that $f \in \mathcal{M}(\alpha, \beta)$ if and only if $f \prec h$.*

The following theorem is a result of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [1], [2], [3])

Theorem 2.1 *Let h convex in $\mathcal{U}$ and $Re[\beta h(z) + \gamma] > 0$, $z \in \mathcal{U}$. If $p \in \mathcal{H}(\mathcal{U})$ with $p(0) = h(0)$ and p satisfied the Briot-Bouquet differential subordination $p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} < h(z)$, then $p(z) < h(z)$.*

3 Main results

Theorem 3.1 *if $F(z) \in \mathcal{M}(\alpha, \beta)$, with $0 \leq \beta < 1$ and $\alpha > 1$, then $f(z) = L_a F(z) \in \mathcal{M}(\alpha, \beta)$, with $0 \leq \beta < 1$ and $\alpha > 1$.*

Proof. By differentiating (1) we obtain:

$$(1 + a)F(z) = af(z) + zf'(z). \tag{3}$$

From (3) we obtain

$$(1 + a)F'(z) = (a + 1)f'(z) + zf''(z). \tag{4}$$

Thus

$$\frac{zF'(z)}{F(z)} = \frac{z(1+a)F'(z)}{(1+a)F(z)} =$$

(5) $$= \frac{z[(a+1)f'(z) + zf''(z)]}{af(z) + zf'(z)} = \frac{(a+1)\dfrac{zf'(z)}{f(z)} + z^2\dfrac{f''(z)}{f(z)}}{a + z\dfrac{f'(z)}{f(z)}}.$$

With notation $\frac{zf'(z)}{f(z)} = p(z)$, where $p(0) = 1$, we have

$$\frac{z^2 f''}{f(z)} = p^2 + zp'(z) - p(z)$$

Thus from (5) we obtain:

(6) $$\frac{zF'(z)}{F(z)} = p(z) + \frac{1}{a+p(z)} \cdot zp'(z)$$

Now, from Remark 2.2 and from hypothesis we have $p(z) + \frac{1}{a+p(z)} \cdot zp'(z) < h(z)$. Also, we have from hypothesis we obtain $Re[p(z) + a] > 0$. IN this condition from theorem 2.1 we obtain $p(z) < h(z)$ and thus from Remark 2.2 we have $f(z) = L_a F(z) \in \mathcal{M}(\alpha, \beta)$, with $0 \leq \beta < 1$ and $\alpha > 1$.

Remark 3.1 *If we consider $\beta = 0$ and $\alpha \to \infty$ in theorem 3.1 we obtain the well now fact that the integral operator L_a, defined by (1), preserve the class of starlike functions.*

References

[1] S.S. Miller, P.T. Mocanu, *Differential subordinations and univalent functions*, Mich. Math., 28 (1981), 157-171.

[2] S.S. Miller, P.T. Mocanu, *Univalent solution of Briot-Bouquet differential equations*, J. Diffrential Equations, 56 (1985), 297-308.

[3] S.S. Miller, P.T. Mocanu, *On some classes of first-order differential subordinations*, Mich. Math., 32 (1985), 185-195.

[4] S. Owa, J. Nishiwaki, *Coefficient estimates for certain classes of analytic functions*, Journal of Inequalities in Pure an Applied Mathematics, Vol.3, Issue 5, Article 72, 2002.

[5] S. Owa, H.M. Srivastawa, *Some generalized convolution properties associated with certain subclasses of analytic functions*, Journal of Inequalities in Pure an Applied Mathematics, Vol.3, Issue 3, Article 42, 2002.

[6] B.A. Uralegaddi, M.D. Ganini, S.M. Sarangi, *Univalent functions with pozitive coefficients*, Tamkang J. Math., 25 (1994), 225-230.

Department of Mathematics
University "Lucian Blaga" of Sibiu,
Str. Dr. I. Ratiu Nr.7
2400 Sibiu, Romania