Georgian Mathematical Journal
1(1994), No. 1, 47-52

MEASURES OF CONTROLLABILITY

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Abstract. We introduce here a new notion, the measure of *controllability* aimed at expressing that one system is "more controllable" than another one. First estimates are given.

1. Introduction

Let Ω be an open set in $\mathbb{R}^n$, bounded or not, with boundary Γ, smooth or not.

In the domain Ω and for $t > 0$, we consider the system whose state $y : y(x,t) = y(x,t;v)$ is given as follows:

$$\frac{\partial y}{\partial t} + Ay = v(x,t)\chi_{\mathcal{O}} \quad \text{in} \quad \Omega \times \{t > 0\}, \tag{1.1}$$

where A = second order elliptic operator in Ω (its coefficients are not necessarily smooth and they may depend on t),

$\mathcal{O}$ = open set $\subset \Omega$,

$\chi_{\mathcal{O}}$ = characteristic function of $\mathcal{O}$,

$v = v(x,t)$ = control function.

We add to (1.1) the *initial* and boundary *conditions* respectively given by

$$y(x,0) = y^0(x) \quad \text{in} \quad \Omega, \quad y^0 \text{ given in } L^2(\Omega), \tag{1.2}$$

and

$$y = 0 \quad \text{on} \quad \Gamma \times \{t > 0\}. \tag{1.3}$$

Under reasonable conditions on the coefficients of A (cf.for instance J.L.Lions [3]), and assuming that

$$v \in L^2(\mathcal{O} \times (0,T)), \tag{1.4}$$

1991 *Mathematics Subject Classification.* 49J20.

equations (1.1),(1.2),(1.3) admit a *unique solution* y, which is such that

$$y,\ \frac{\partial y}{\partial x_i} \in L^2(\Omega \times (0,T)). \tag{1.5}$$

This defines the state of the system, with distributed control with support in $\mathcal{O}$. □

Remark 1.1. Boundary condition (1.3) is taken here *to fix ideas*. What follows readily applies to other situations corresponding to other boundary conditions. □

Remark 1.2. All what follows readily extends to higher order parabolic equations, to systems of parabolic equations and actually to *all evolution equations*, provided they are *linear*. This will be reported elsewhere. Cf. also the Remarks of the last section of this paper. □

Remark 1.3. One knows that (J.L.Lions [3]) after a possible change on a set of 0 measure, the function $t \to y(t) = y(\cdot,t)$ is continuous from $[0,T] \to L^2(\Omega)$. □

Approximate controllability is defined as follows (cf. for instance J.L.Lions [4]). We are given T and $y^1 \in L^2(\Omega)$. Let B denote the unit ball in $L^2(\Omega)$ and let β be a positive number arbitrarily small.

It is known (J.L.Lions [5]) that, when v spans $L^2(\mathcal{O} \times (0,T))$, the functions $y(\cdot,T;v)$ describe an affine space in $L^2(\Omega)$ which is dense in $L^2(\Omega)$. Therefore one can always find functions v (controls) such that

$$y(T;v) \in y^1 + \beta B \tag{1.6}$$

and there are *infinitely many v's* such that (1.6) takes place. One says that the system is *approximately controllable*. It is natural to look for the (actually unique) element v such that

$$\frac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 dx\, dt = \min \tag{1.7}$$

where v is restricted to those elements such that (1.6) takes place.

The question we want to address here is the following: *when can we say that a system is more controllable than another one?*

In this question we assume that Ω and that $\mathcal{O}$ do not change. Then the min in (1.7) is a quantity which depends on A, y^0, y^1 and β and T. We write

$$\begin{aligned} &\inf_v \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 dx\, dt = M(A, y^0, y^1, \beta, T), \\ &y(T;v) \in y^1 + \beta B. \end{aligned} \tag{1.8}$$

We have to introduce a quantity which is *independent* of y^0 and of y^1 but which only depends on the sets described by y^0 and by y^1.

We shall assume

$$y^0 \in \alpha_0 B, \quad y^1 \in \alpha_1 B \tag{1.9}$$

and we introduce as a "measure of controllability" the quantity

$$M(A, \alpha_0, \alpha_1, \beta, T) = \sup_{\substack{y^0 \in \alpha_0 B \\ y^1 \in \alpha_1 B}} M(A, y^0, y^1, \beta, T). \tag{1.10}$$

Remark 1.4. This quantity seems to be introduced here for the first time. The study of the function

$$A \to M(A, \alpha_0, \alpha_1, \beta, T) \tag{1.11}$$

leads to many seemingly interesting open questions. We shall return to these questions in other occasions. □

Remark 1.5. It is not obvious that the quantity introduced in (1.9) is always finite. Indeed this quantity is finite iff $\beta > \alpha_1$. □

Remark 1.6. We shall give below a number of simple formulas reducing the number of variables $\alpha_0, \alpha_1, \beta$ to actually one variable. □

We are now going to give a formula for $M(A, \alpha_0, \alpha_1, \beta, T)$ which is based on *duality arguments*.

2. Duality Formula for the Measure of Controllability

We introduce the decomposition

$$y(x, t; v) = y(v) = y_0 + z(v) \tag{2.1}$$

where

$$\begin{aligned} &\frac{\partial y_0}{\partial t} + A y_0 = 0, \\ &y_0(0) = y^0, \quad y_0 = 0 \quad \text{on} \quad \Gamma \times (0, T) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\frac{\partial z}{\partial t} + A z = v \chi_{\mathcal{O}}, \\ &z(0) = 0, \quad z = 0 \quad \text{on} \quad \Gamma \times (0, T). \end{aligned} \tag{2.3}$$

Then

$$\begin{aligned} &M(A, y^0, y^1, \beta, T) = \inf \frac{1}{2} \iint_{\mathcal{O} \times (0,T)} v^2 dx\, dt, \\ &z(T; v) \in y^1 - y_0(T) + \beta B. \end{aligned} \tag{2.4}$$

We introduce the convex functions defined by

$$F_0(v) = \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 dx\, dt, \quad v \in L^2(\mathcal{O}\times(0,T)), \tag{2.5}$$

$$F_1(f) = \begin{cases} 0 & \text{if } f \in y^1 - y_0(T) + \beta B, \\ +\infty & \text{otherwise in } \; L^2(\Omega). \end{cases} \tag{2.6}$$

We define the linear operator L by

$$Lv = z(T; v). \tag{2.7}$$

One has

$$L \in \mathcal{L}(L^2(\mathcal{O}\times(0,T));\ L^2(\Omega)). \tag{2.8}$$

With those notations (this is only a matter of definition)

$$M(A, y^0, y^1, \beta, T) = \inf_{v \in L^2(\mathcal{O}\times(0,T))} F_0(v) + F_1(Lv). \quad \square \tag{2.9}$$

The next step is to use Fenchel-Rockafellar duality (cf. T.R.Rockafellar [6] and the presentation made in I.Ekeland and R.Temam [1]).

In general, the conjugate function F_i^* of F_i is defined by

$$F_i^*(f) = \sup_{\widehat{f}} [(f, \widehat{f}) - F_i(\widehat{f})].$$

With these definitions, one has

$$\begin{aligned} &F_0^*(v) = F_0(v), \\ &F_1^*(f) = (f, y^1 - y_0(T)) + \beta \|f\|, \\ &\text{where } \; \|f\| \in \Big(\int_\Omega f^2 dx\Big)^{\frac{1}{2}}. \end{aligned} \tag{2.10}$$

Let L^* denote the adjoint of L. Then (T.R.Rockafellar, loc.cit.)

$$\begin{aligned} &\inf_{v \in L^2(\mathcal{O}\times(0,T))} F_0(v) + F_1(Lv) = \\ &- \inf_{f \in L^2(\Omega)} F_0^*(L^* f) + F_1^*(-f). \ \square \end{aligned} \tag{2.11}$$

The operator L^* is given as follows. If f is given in $L^2(\Omega)$, we solve

$$\begin{aligned} &-\frac{\partial \psi}{\partial t} + A^* \psi = 0, \quad t < T, \\ &\psi(x, T) = f(x) \quad \text{in} \quad \Omega, \\ &\psi = 0 \quad \text{on} \quad \Gamma \times \{t < T\}, \end{aligned} \tag{2.12}$$

where A^* = adjoint of A.

This problem admits a unique solution $\psi(x,t) = \psi(x,t;f) = \psi(f)$.

Then one easily verifies that

$$L^* f = \psi \chi_{\mathcal{O}}. \tag{2.13}$$

Using this result, (2.11), and (2.10), we obtain

$$\begin{aligned} M(A, y^0, y^1, \beta, T) = - \inf_{f \in L^2(\Omega)} \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} & \psi^2 dx\, dt - \\ & - (f, y^1 - y_0(T)) + \beta \|f\|. \end{aligned} \tag{2.14}$$

If we multiply (2.12) by y_0, we obtain after integration by parts

$$-(f, y_0(T)) + (\psi(0), y^0) = 0 \tag{2.15}$$

so that (2.14) can be written

$$\begin{aligned} & M(A, y^0, y^1, \beta, T) = \\ & = - \inf_{f \in L^2(\Omega)} \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} v^2 dx\, dt - \\ & - (f, y^1) + (\psi(0), y^0) + \|f\|. \quad \square \end{aligned} \tag{2.16}$$

By definition

$$\begin{aligned} & M(A, \alpha_0, \alpha_1, \beta, T) = \\ & = \sup_{y^0 \in \alpha_0 B, y^1 \in \alpha_1 B} M(A, y^0, y^1, \beta, T) = (\text{using (2.16)}) = \\ & = - \inf_{y^0 \in \alpha_0 B, y^1 \in \alpha_1 B, f \in L^2(\Omega)} \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} \psi^2 dx\, dt - \\ & - (f, y^1) + (\psi(0), y^0) + \beta \|f\|, \end{aligned} \tag{2.17}$$

i.e.

$$\begin{aligned} & M(A, \alpha_0, \alpha_1, \beta, T) = \\ & = \inf_f \big[\frac{1}{2} \iint_{\mathcal{O}\times(0,T)} \psi^2 dx\, dt + \\ & + (\beta - \alpha_1)\|f\| - \alpha_0 \|\psi(0)\| \big]. \end{aligned} \tag{2.18}$$

In summary:

the measure of controllability is given by formula (2.18), *where* $\psi = \psi(f)$ *is given by* (2.12). $\square$ (2.19)

Remark 2.1. One can show that the $\inf_f$ in (2.18) is finite iff $\beta > \alpha_1$. $\square$

One has

$$M(A,\alpha_0,\alpha_1,\beta,T) = M(A,\alpha_0,0,\beta-\alpha_1,T), \quad \beta > \alpha_1. \tag{2.20}$$

Therefore it suffices to consider the following situation:

$$\sup_{y^0\in\alpha\beta} \inf_{y(T;v)\in\beta B} \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} v^2 dx\,dt = M_0(A,\alpha,\beta,T), \tag{2.21}$$

(Then $M(A,\alpha_0,\alpha_1,\beta,T) = M_0(A,\alpha_0,\beta-\alpha_1,T)$).

One verifies directly that

$$M_0(A,\alpha,\beta,T) = \alpha^2 M_0(A,1,\frac{\beta}{\alpha},T), \tag{2.22}$$

$$M_0(A,\alpha,\beta,T) = \begin{cases} 0 \quad \text{for } \beta \text{ large enough}, \\ \text{increases to } +\infty \text{ as } \beta \text{ decreases to } 0. \end{cases} \tag{2.23}$$

Remark 2.2. Formula (2.18) is constructive. One can deduce from it numerical algorithms for the approximation of M. Cf. R.Glowinski and J.L.Lions [2]. □

References

1. I.Ekeland and R.Temam, Analyse Convexe et problèmes variationnels. *Dunod, Gauthier Villars, Paris-Bruxeles-Montréal*, 1974.

2. R.Glowinski and J.L.Lions, To appear in *Acta Numerica,* 1993.

3. J.L.Lions, Equations Differentielles Opérationnelles et Problèmes aux limites. *Springer-Verlag, Berlin-New York*, 1961.

4. J.L.Lions, Exact Controllability for distributed systems. Some trends and some problems. *Applied and Industrial Mathematics, R.Spigler (ed.),* 59-84, *Kluwer,* 1991.

5. ——, Controle Optimal de systèmes gouvernés par les équations aux dérivées partielles. *Dunod, Gauthier-Villars, Paris,* 1968.

6. T.R.Rockafellar, Duality and stability in extremum problems involving convex functions. *Pac. J. Math.* **21**(1967), 167-187.

(Received 26.11.1992)

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