Georgian Mathematical Journal
1(1994), No. 3, 325-341

CONTACT PROBLEMS FOR TWO ANISOTROPIC HALF-PLANES WITH SLITS

SH. ZAZASHVILI

Abstract. The problem of a stressed state in a nonhomogeneous infinite plane consisting of two different anisotropic half-planes and having slits of the finite number on the interface line is investigated. It is assumed that a difference between the displacement and stress vector values is given on interface line segments; on the edges of slits we have the following data: boundary values of the stress vector (problem of stress) or displacement vector values on the one side of slits, and stress vector values on the other side (mixed problem). Solutions are constructed in quadratures.

In this paper, employing the methods of the potential theory and of systems of singular integral equations, we investigate problems of a stressed state in a nonhomogeneous infinite plane consisting of two anisotropic half-planes with different elastic constants and having slits on the interface line between the half-planes. The stressed state is determined giving displacement and stress vector jumps on the interface line segments, and boundary values of either the stress vector or of the displacement vector on the one side of slits, and stress vector values on the other side.

The difficulty of solving the problems lies, in particular, in lengthy calculations one has to perform in order to verify certain conditions, but it can be overcome using the constants introduced by of M.O. Basheleishvili [1]. The solutions obtained are constructed in quadratures.

It should be observed that, when the half-planes are welded to each other along the interface line segments, the problem of stress was studied on the basis of the theory of functions of a complex variable in [2], where the problem is reduced to the solution of four problems of linear conjugation. The solution of the problem is not, however, simple and demands some refinement.

Formulation of Problems. Let the real x-axis be the interface line between two different anisotropic materials filling up the upper ($y > 0$) and

1991 *Mathematics Subject Classification.* 73C30.

the lower ($y<0$) half-planes and having Hook's coefficients

$$A_{11}^{(0)},\quad A_{12}^{(0)},\quad A_{22}^{(0)},\quad A_{13}^{(0)},\quad A_{23}^{(0)},\quad A_{33}^{(0)}$$

and

$$A_{11}^{(1)},\quad A_{12}^{(1)},\quad A_{22}^{(1)},\quad A_{13}^{(1)},\quad A_{23}^{(1)},\quad A_{33}^{(1)}$$

respectively.

It is assumed that slits are located on the segments $l_p = a_p b_p$, $p = 1,2,\dots,n$, of the x-axis. Let $l = \cup_{p=1}^{n} l_p$ and L be the remainder part of the real axis outside slits. Denote the domain $y>0$ by D_0, and the domain $y<0$ by D_1.

As is known, in the absence of mass force the system of differential equations of equilibrium of an anisotropic elastic body in the domain D_j, $j=0,1$, looks like [3]

$$\begin{aligned}
&A_{11}^{(j)}\frac{\partial^2 u^{(j)}}{\partial x^2} + 2A_{13}^{(j)}\frac{\partial^2 u^{(j)}}{\partial x\partial y} + A_{33}^{(j)}\frac{\partial^2 u^{(j)}}{\partial y^2} + A_{13}^{(j)}\frac{\partial^2 v^{(j)}}{\partial x^2} + \\
&\qquad + (A_{12}^{(j)} + A_{33}^{(j)})\frac{\partial^2 v^{(j)}}{\partial x\partial y} + A_{23}^{(j)}\frac{\partial^2 v^{(j)}}{\partial y^2} = 0,\\
&A_{13}^{(j)}\frac{\partial^2 u^{(j)}}{\partial x^2} + (A_{12}^{(j)} + A_{33}^{(j)})\frac{\partial^2 u^{(j)}}{\partial x\partial y} + A_{23}^{(j)}\frac{\partial^2 u^{(j)}}{\partial y^2} + A_{33}^{(j)}\frac{\partial^2 v^{(j)}}{\partial x^2} + \\
&\qquad + 2A_{23}^{(j)}\frac{\partial^2 v^{(j)}}{\partial x\partial y} + A_{22}^{(j)}\frac{\partial^2 v^{(j)}}{\partial y^2} = 0,
\end{aligned}\tag{1}$$

where $u^{(j)}$ and $v^{(j)}$ are the Cartesian coordinates of the displacement vector.

The stressed state in the anisotropic body occupying the domain D_j is determined by three stress components $\sigma_x^{(j)}$, $\sigma_y^{(j)}$, $\tau_{xy}^{(j)}$, which, in turn, are expressed by means of the strain components $\varepsilon_x^{(j)}$, $\varepsilon_y^{(j)}$, $\varepsilon_{xy}^{(j)}$ as follows:

$$\begin{aligned}
\sigma_x^{(j)} &= A_{11}^{(j)}\varepsilon_x^{(j)} + A_{12}^{(j)}\varepsilon_y^{(j)} + A_{13}^{(j)}\varepsilon_{xy}^{(j)},\\
\sigma_y^{(j)} &= A_{12}^{(j)}\varepsilon_x^{(j)} + A_{22}^{(j)}\varepsilon_y^{(j)} + A_{23}^{(j)}\varepsilon_{xy}^{(j)},\\
\tau_{xy}^{(j)} &= A_{13}^{(j)}\varepsilon_x^{(j)} + A_{23}^{(j)}\varepsilon_y^{(j)} + A_{33}^{(j)}\varepsilon_{xy}^{(j)},
\end{aligned}\tag{2}$$

where

$$\varepsilon_x^{(j)} = \frac{\partial u^{(j)}}{\partial x},\quad \varepsilon_y^{(j)} = \frac{\partial v^{(j)}}{\partial y},\quad \varepsilon_{xy}^{(j)} = \frac{\partial u^{(j)}}{\partial y} + \frac{\partial v^{(j)}}{\partial x}.$$

We will consider the following boundary-contact problems:

In the domain D_j, $j=0,1$, find a regular solution of system (1), i.e., determine the displacement components $u^{(j)}$, $v^{(j)}$ and the stress components

$\sigma_x^{(j)}$, $\sigma_y^{(j)}$, $\tau_{xy}^{(j)}$ when on L a difference is given between boundary values of the displacement and stress vectors[1]

$$\begin{pmatrix} u^{(0)} \\ v^{(0)} \end{pmatrix}^+ - \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix}^- = \mathbf{f}, \quad \begin{pmatrix} \tau_{xy}^{(0)} \\ \sigma_y^{(0)} \end{pmatrix}^+ - \begin{pmatrix} \tau_{xy}^{(1)} \\ \sigma_y^{(1)} \end{pmatrix}^- = -\boldsymbol{\varphi}, \tag{3}$$

while on the edges of slits we have either boundary values of the stress vectors

$$\begin{pmatrix} \tau_{xy}^{(0)} \\ \sigma_y^{(0)} \end{pmatrix}^+ = -\mathbf{F}, \quad \begin{pmatrix} \tau_{xy}^{(1)} \\ \sigma_y^{(1)} \end{pmatrix}^- = -\Phi, \tag{4}$$

or boundary values of the stress vector from D_0 and of the displacement vector from D_1

$$\begin{pmatrix} \tau_{xy}^{(0)} \\ \sigma_y^{(0)} \end{pmatrix}^+ = -\mathbf{F}, \quad \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix}^- = \Phi. \tag{5}$$

The regularity of the solution of system (1) implies that: 1) this solution has continuous partial derivatives of second order in the domain D_j, $j = 0, 1$; 2) it can be continuously extended onto the whole real axis; 3) stress components $\sigma_x^{(j)}$, $\sigma_y^{(j)}$, $\tau_{xy}^{(j)}$ which by (2) correspond to it can be continuously extended onto the whole real axis except perhaps the end points of slits in whose neighbourhoods they have an integrable singularity.

In the sequel the problem with the boundary conditions (3), (4) will be called the *problem of stress*, while the problem with the boundary conditions (3), (5) the *mixed problem.* We will investigate each of these problems separately.

Problem of Stress. It is assumed that the known vectors $\mathbf{f}$, $\boldsymbol{\varphi}$, $\mathbf{F}$ and Φ satisfy the following conditions:

a) when $|x| \to \infty$

$$|x|^\alpha \mathbf{f}(x) \to \boldsymbol{\beta}, \quad |x|^{1+\delta} \boldsymbol{\varphi}(x) \to \boldsymbol{\gamma}_0 \quad (\alpha > 0,\ \delta > 0), \tag{6}$$

where $\boldsymbol{\beta}$, γ_0 are the constant vectors;

b) $\mathbf{f}$ belongs to the Hölder class on L (including the neighbourhood of the point at infinity);

c) $\mathbf{f}'$, $\boldsymbol{\varphi}$, $\mathbf{F}$ and Φ belong to the class H^* [4];

d) the vector $\mathbf{f}$ satisfies the conditions

$$\mathbf{f}(a_p) = \mathbf{f}(b_p) = 0, \quad p = 1, 2, \dots, n; \tag{7}$$

[1] All the vectors considered are columns but they will sometimes be written as rows. To make the formulas shorter, the commonly used notations $\frac{1}{a}\mathbf{C}$ and $b\mathbf{C}$ with numbers $a \neq 0$, b and vector or matrix $\mathbf{C}$ are often replaced by $\frac{\mathbf{C}}{a}$ and $\mathbf{C}b$, respectively. The superscript $+(-)$ denotes that the boundary value of the function is taken from $D_0\,(D_1)$.

e) stress and rotation vanish at infinity.

Like in [5], the displacement vector $(u^{(j)}, v^{(j)})$ in the domain D_j, $j = 0, 1$, will be sought as a combination of simple- and double-layer potentials

$$\begin{pmatrix} u^{(j)} \\ v^{(j)} \end{pmatrix} = \frac{1}{\pi} \mathrm{Im} \sum_{k=1}^{2} \mathbf{E}^{(j)}(k) \Big\{ (\mathbf{A}^{(j)} + i\mathbf{B}^{(j)}) \Big(\int_L \frac{\mathbf{f}(t)\,dt}{t - z_{kj}} + \\ + \int_l \mathbf{g}(t) \ln(t - z_{kj})\,dt \Big) - (\mathbf{C}^{(j)} + i\mathbf{D}^{(j)}) \int_{-\infty}^{+\infty} \mathbf{h}(t) \ln(t - z_{kj})\,dt \Big\} - \\ - \frac{1}{\pi} \mathrm{Im} \sum_{k=1}^{2} \mathbf{A}^{(j)}(k) \mathbf{X}^{(j)} \mathbf{P} \ln(z_{kj} + i(-1)^j), \tag{8}$$

where $\mathbf{g} = (g_1, g_2)$ and $\mathbf{h} = (h_1, h_2)$ are the unknown vectors which , in view of the fact that the logarithmic function is many-valued, must satisfy the conditions

$$\int_{l_p} \mathbf{g}(t)\,dt = 0, \quad p = 1, 2, \dots, n, \tag{9}$$

$$\int_{-\infty}^{+\infty} \mathbf{h}(t)\,dt = 0; \tag{10}$$

$\mathbf{P}$ denotes the principal vector of external force

$$\mathbf{P} = \int_{-\infty}^{+\infty} \left\{ \begin{pmatrix} \tau_{xy}^{(1)} \\ \sigma_y^{(1)} \end{pmatrix}^{-} - \begin{pmatrix} \tau_{xy}^{(0)} \\ \sigma_y^{(0)} \end{pmatrix}^{+} \right\} dt; \tag{11}$$

$i = \sqrt{-1}$ is the complex unity; $z_{kj} = x + \alpha_{kj} y$, $k = 1, 2$, $j = 0, 1$, where $\alpha_{kj} = a_{kj} + i b_{kj}$ $(b_{kj} > 0)$ is the root of the characteristic equation corresponding to system (1) (as known [3], the latter equation is a fourth-degree equation with real coefficients of the form

$$a_{11}^{(j)} \alpha_j^4 - 2a_{13}^{(j)} \alpha_j^3 + (2a_{12}^{(j)} + a_{33}^{(j)}) \alpha_j^2 - 2a_{23}^{(j)} \alpha_j + a_{22}^{(j)} = 0$$

and has complex roots only, $a_{ks}^{(j)}$ are the coefficients at $\sigma_x^{(j)}$, $\sigma_y^{(j)}$ and $\tau_{xy}^{(j)}$ when the strain components from (2) are expressed in terms of stress components); the constant two-dimensional matrices $\mathbf{A}^{(j)}(k)$ and $\mathbf{E}^{(j)}(k)$, $k = 1, 2$, $j = 0, 1$, occur in [1] when constructing the matrix of fundamental solutions of system (1) and the double-layer potential, respectively, and are written as

$$\mathbf{A}^{(j)}(k) = \begin{Vmatrix} A_k^{(j)} & B_k^{(j)} \\ B_k^{(j)} & C_k^{(j)} \end{Vmatrix}, \quad \mathbf{E}^{(j)}(k) = -\frac{i}{m_j} A_k^{(j)} \begin{Vmatrix} B_j & -A_j \\ -A_j & C_j \end{Vmatrix},$$

$$\sum_{k=1}^{2}\mathbf{E}^{(j)}(k)=\mathbf{E}=\begin{Vmatrix}1&0\\0&1\end{Vmatrix},$$
$$A_k^{(j)}=-\frac{2}{\Delta^{(j)}a_{11}^{(j)}}\{A_{22}^{(j)}\alpha_{kj}^2+2A_{23}^{(j)}\alpha_{kj}+A_{33}^{(j)}\}d_{kj},$$
$$C_k^{(j)}=-\frac{2}{\Delta^{(j)}a_{11}^{(j)}}\{A_{33}^{(j)}\alpha_{kj}^2+2A_{13}^{(j)}\alpha_{kj}+A_{11}^{(j)}\}d_{kj},$$
$$B_k^{(j)}=\frac{2}{\Delta^{(j)}a_{11}^{(j)}}\{A_{23}^{(j)}\alpha_{kj}^2+(A_{12}^{(j)}+A_{33}^{(j)})\alpha_{kj}+A_{13}^{(j)}\}d_{kj},$$
$$\Delta^{(j)}a_{11}^{(j)}=A_{22}^{(j)}A_{33}^{(j)}-(A_{12}^{(j)})^2>0,$$
$$d_{1j}^{-1}=(\alpha_{1j}-\bar\alpha_{1j})(\alpha_{1j}-\alpha_{2j})(\alpha_{1j}-\bar\alpha_{2j}),$$
$$d_{2j}^{-1}=(\alpha_{2j}-\alpha_{1j})(\alpha_{2j}-\bar\alpha_{1j})(\alpha_{2j}-\bar\alpha_{2j}),$$

$\Delta^{(j)}$ is the determinant of system (2) whose positiveness follows from the positiveness of potential energy; A_j, B_j, C_j, ω_j, m_j, $\varkappa_N^{(j)}$ are the above-mentioned Basheleishvili's constants [1], [5]:

$$A_j=2i\sum_{k=1}^{2}d_{kj},\quad B_j=2i\sum_{k=1}^{2}\alpha_{kj}^2d_{kj},\quad C_j=2i\sum_{k=1}^{2}\alpha_{kj}d_{kj},$$
$$\omega_j=b_{1j}b_{2j}-a_{1j}a_{2j}+\frac{a_{13}^{(j)}}{a_{11}^{(j)}},\quad m_j=a_{11}^{(j)}\big[1-\omega_j^2(B_jC_j-A_j^2)\big],$$
$$\varkappa_N^{(j)}=\frac{\omega_j(B_jC_j-A_j^2)}{m_j},$$

they satisfy the conditions

$$\operatorname{Im}A_j=0,\quad B_j>0,\quad C_j>0,\quad B_jC_j-A_j^2>0,$$
$$m_j>0,\quad \varkappa_N^{(j)}>0,\quad \omega_ja_{11}^{(j)}>0,\quad j=0,1;$$

the constant matrices $\mathbf{A}^{(j)}$, $\mathbf{B}^{(j)}$, $\mathbf{C}^{(j)}$, $\mathbf{D}^{(j)}$, $\mathbf{X}^{(j)}$ in (8) ensure the fulfilment of the contact conditions (3) and have the form [5]

$$\mathbf{A}^{(0)}=\frac{1}{\Delta}\Bigg\{\Big(\frac{B_1C_1-A_1^2}{m_1a_{11}^{(1)}}+\varkappa_N^{(0)}\varkappa_N^{(1)}\Big)\mathbf{E}+$$
$$+\frac{1}{m_0m_1}\begin{Vmatrix}B_1C_0-A_0A_1&A_0C_1-A_1C_0\\A_0B_1-A_1B_0&B_0C_1-A_0A_1\end{Vmatrix}\Bigg\},$$
$$\mathbf{B}^{(0)}=-\frac{1}{\Delta}\left\{\frac{\varkappa_N^{(1)}}{m_0}\begin{Vmatrix}A_0&-C_0\\B_0&-A_0\end{Vmatrix}+\frac{\varkappa_N^{(0)}}{m_1}\begin{Vmatrix}A_1&-C_1\\B_1&-A_1\end{Vmatrix}\right\},$$

$$\mathbf{C}^{(0)} = \frac{\varkappa_N^{(0)} - \varkappa_N^{(1)}}{\Delta} \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix},$$

$$\mathbf{D}^{(0)} = \frac{1}{\Delta}\left\{\frac{1}{m_0}\begin{Vmatrix} C_0 & A_0 \\ A_0 & B_0 \end{Vmatrix} + \frac{1}{m_1}\begin{Vmatrix} C_1 & A_1 \\ A_1 & B_1 \end{Vmatrix}\right\},$$

$$\mathbf{X}^{(0)} = \frac{m_1}{\Delta^*(B_1C_1 - A_1^2)}\Big\{m_1\mathbf{E} + \\ + \frac{m_0}{B_0C_0 - A_0^2}\begin{Vmatrix} B_0C_1 - A_0A_1 & A_1B_0 - A_0B_1 \\ A_1C_0 - A_0C_1 & B_1C_0 - A_0A_1 \end{Vmatrix}\Big\},$$

where

$$\Delta = \frac{B_0C_0 - A_0^2}{m_0 a_{11}^{(0)}} + \frac{B_1C_1 - A_1^2}{m_1 a_{11}^{(1)}} + \\ + \frac{B_1C_0 + B_0C_1 - 2A_0A_1}{m_0m_1} + 2\varkappa_N^{(0)}\varkappa_N^{(1)} > 0,$$

$$\Delta^* = \frac{m_0^2}{B_0C_0 - A_0^2} + \frac{m_1^2}{B_1C_1 - A_1^2} + \\ + \frac{m_0m_1(B_1C_0 + B_0C_1 - 2A_0A_1)}{(B_0C_0 - A_0^2)(B_1C_1 - A_1^2)} > 0,$$

the matrices $\mathbf{A}^{(1)}$, $\mathbf{B}^{(1)}$, $\mathbf{C}^{(1)}$, $\mathbf{D}^{(1)}$, $\mathbf{X}^{(1)}$ are obtained from $\mathbf{A}^{(0)}$, $\mathbf{B}^{(0)}$, $\mathbf{C}^{(0)}$, $\mathbf{D}^{(0)}$, $\mathbf{X}^{(0)}$ permuting the indices 0 and 1.

It is easy to obtain

$$\mathbf{A}^{(0)} + \mathbf{A}^{(1)} = \mathbf{E}, \quad \mathbf{X}^{(0)} + \mathbf{X}^{(1)} = \mathbf{E},$$

$$\frac{m_1}{B_1C_1 - A_1^2}\begin{Vmatrix} C_1 & A_1 \\ A_1 & B_1 \end{Vmatrix}\mathbf{X}^{(1)} = \frac{m_0}{B_0C_0 - A_0^2}\begin{Vmatrix} C_0 & A_0 \\ A_0 & B_0 \end{Vmatrix}\mathbf{X}^{(0)}.$$

By virtue of conditions (7) the stress components $\tau_{xy}^{(j)}$ and $\sigma_y^{(j)}$ will have the form

$$\begin{pmatrix} \tau_{xy}^{(j)} \\ \sigma_y^{(j)} \end{pmatrix} = \frac{1}{\pi}\mathrm{Im}\sum_{k=1}^{2}\mathbf{E}_0^{(j)}(k)\bigg\{(\mathbf{A}^{(j)} + i\mathbf{B}^{(j)})\Big(\int_l \frac{\mathbf{g}(t)\,dt}{t - z_{kj}} - \\ - \int_L \frac{\mathbf{f}'(t)\,dt}{t - z_{kj}}\Big) - (\mathbf{C}^{(j)} + i\mathbf{D}^{(j)})\int_{-\infty}^{+\infty}\frac{\mathbf{h}(t)\,dt}{t - z_{kj}}\bigg\} + \\ + \frac{1}{\pi}\mathrm{Im}\sum_{k=1}^{2}\mathbf{N}^{(j)}(k)\mathbf{X}^{(j)}\frac{\mathbf{P}}{z_{kj} + i(-1)^j}, \quad j = 0, 1, \tag{12}$$

where for $k = 1, 2$, $j = 0, 1$

$$\mathbf{E}_0^{(j)}(k) = -\frac{i}{m_j}\mathbf{N}^{(j)}(k)\begin{Vmatrix} B_j & -A_j \\ -A_j & C_j \end{Vmatrix},$$

$$\mathbf{N}^{(j)}(k) = \left(\varkappa_N^{(j)} \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix} - \frac{i}{m_j} \begin{Vmatrix} B_j & -A_j \\ -A_j & C_j \end{Vmatrix}\right) \mathbf{A}^{(j)}(k),$$

$$\sum_{k=1}^{2} \mathbf{E}_0^{(j)}(k) = \varkappa_N^{(j)} \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix} - \frac{i}{m_j} \begin{Vmatrix} B_j & -A_j \\ -A_j & C_j \end{Vmatrix},$$

$$\sum_{k=1}^{2} \mathbf{N}^{(j)}(k) = \mathbf{E} + i\omega_j \begin{Vmatrix} -A_j & -B_j \\ C_j & A_j \end{Vmatrix}.$$

It is easy to show that the vector $(u^{(j)}, v^{(j)})$ given by formula (8) satisfies the first of the contact conditions (3).

If we now calculate the boundary values of the vector $(\tau_{xy}^{(j)}, \sigma_y^{(j)})$, $j = 0, 1$, by formula (12), then we readily obtain that at points of the real axis, except perhaps points a_p, b_p, $p = 1, 2, \ldots, n$,

$$\mathbf{h}(x) = \begin{pmatrix} \tau_{xy}^{(1)} \\ \sigma_y^{(1)} \end{pmatrix}^{-} - \begin{pmatrix} \tau_{xy}^{(0)} \\ \sigma_y^{(0)} \end{pmatrix}^{+} - \frac{1}{\pi} \operatorname{Im} \frac{\sum_{k=1}^{2} \mathbf{N}^{(1)}(k)}{x - i} \mathbf{X}^{(1)} \mathbf{P} + \\ + \frac{1}{\pi} \operatorname{Im} \frac{\sum_{k=1}^{2} \mathbf{N}^{(0)}(k)}{x + i} \mathbf{X}^{(0)} \mathbf{P}, \tag{13}$$

Since the difference

$$\begin{pmatrix} \tau_{xy}^{(1)} \\ \sigma_y^{(1)} \end{pmatrix}^{-} - \begin{pmatrix} \tau_{xy}^{(0)} \\ \sigma_y^{(0)} \end{pmatrix}^{+}$$

is known due to conditions (3) and (4), the vector $\mathbf{h}$ will be known, too. One can easily verify that $\mathbf{h}$ defined by equality (13) will satisfy condition (10).

Therefore the unknown vector $\mathbf{h}$ is defined by equality (13) and there remains for us to define the vector $\mathbf{g}$.

By summing up the boundary values of the vector $(\tau_{xy}^{(j)}, \sigma_y^{(j)})$ from D_0 and D_1 at points belonging to l, except perhaps end points, we obtain the following system of integral equations for the vector $\mathbf{g}$ [6]:

$$\mathbf{A}^* g(x) + \frac{\mathbf{B}^*}{\pi} \int_l \frac{\mathbf{g}(t)\, dt}{t - x} = \mathbf{\Omega}(x), \quad x \in l, \tag{14}$$

where

$$\mathbf{A}^* = \Big(\frac{\varkappa_N^{(1)}(B_0 C_0 - A_0^2)}{m_0 a_{11}^{(0)}} - \frac{\varkappa_N^{(0)}(B_1 C_1 - A_1^2)}{m_1 a_{11}^{(1)}}\Big) \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix},$$

$$\mathbf{B}^* = \frac{B_1 C_1 - A_1^2}{m_0 m_1 a_{11}^{(1)}} \begin{Vmatrix} B_0 & -A_0 \\ -A_0 & C_0 \end{Vmatrix} + \frac{B_0 C_0 - A_0^2}{m_0 m_1 a_{11}^{(1)}} \begin{Vmatrix} B_1 & -A_1 \\ -A_1 & C_1 \end{Vmatrix},$$

$$\begin{aligned}
&\mathbf{\Omega}(x)=\frac{\Delta}{2}\big(\mathbf{F}(x)+\Phi(x)\big)+\frac{\mathbf{B}^*}{\pi}\int_L\frac{\mathbf{f}'(t)\,dt}{t-x}-\frac{1}{2}\mathbf{C}^*\mathbf{h}(x)-\\
&\quad-\frac{\mathbf{D}^*}{\pi}\int_{-\infty}^{+\infty}\frac{\mathbf{h}(t)\,dt}{t-x}+\frac{\Delta}{2\pi}\mathrm{Im}\frac{\sum_{k=1}^2\mathbf{N}^{(0)}(k)}{x+i}\mathbf{X}^{(0)}\mathbf{P}+\\
&\quad+\frac{\Delta}{2\pi}\mathrm{Im}\frac{\sum_{k=1}^2\mathbf{N}^{(1)}(k)}{x-i}\mathbf{X}^{(1)}\mathbf{P},\\
&\mathbf{C}^*=\Big(\frac{B_0C_0-A_0^2}{m_0a_{11}^{(0)}}+\frac{B_1C_1-A_1^2}{m_1a_{11}^{(1)}}\Big)\mathbf{E}+\\
&\quad+\frac{1}{m_0m_1}\begin{Vmatrix}B_0C_1-C_0B_1 & 2(A_1B_0-A_0B_1)\\ 2(A_1C_0-A_0C_1) & B_1C_0-B_0C_1\end{Vmatrix},\\
&\mathbf{D}^*=\frac{\varkappa_N^{(0)}}{m_1}\begin{Vmatrix}-A_1 & -B_1\\ C_1 & A_1\end{Vmatrix}+\frac{\varkappa_N^{(1)}}{m_0}\begin{Vmatrix}-A_0 & -B_0\\ C_0 & A_0\end{Vmatrix}.
\end{aligned}$$

The properties of the boundary data enable us to conclude that $\mathbf{\Omega}=(\Omega_1,\Omega_2)$ is a vector of the class H^*. The solution of system (14) should be sought for in the same class.

System (14) is of the normal type, since

$$\det(\mathbf{A}^*\pm i\mathbf{B}^*)=-\frac{(B_0C_0-A_0^2)(B_1C_1-A_1^2)}{m_0m_1a_{11}^{(0)}a_{11}^{(1)}}\Delta<0.$$

As known, in construction the solution of system (14) we encounter certain difficulties and hence we have to seek for its solution by reducing it to a singular integral equation for some scalar function [7].

To this end, multiplying the first equation of system (14) by the constant M and adding to the second equation, we obtain

$$\begin{aligned}
S\big(g_1(x)-Mg_2(x)\big)+\frac{1}{\pi}\int_l\frac{(B_{11}^*M+B_{21}^*)g_1(t)+(B_{12}^*M+B_{22}^*)g_2(t)}{t-x}\,dt=\\
=\Omega_2(x)+M\Omega_1(x),
\end{aligned}\tag{15}$$

where B_{kj}^*, $k=1,2$, $j=1,2$, are the elements of the matrix $\mathbf{B}^*$ and

$$S=\frac{\varkappa_N^{(1)}(B_0C_0-A_0^2)}{m_0a_{11}^{(0)}}-\frac{\varkappa_N^{(0)}(B_1C_1-A_1^2)}{m_1a_{11}^{(1)}}.$$

Next we choose a constant M such that

$$B_{12}^*M+B_{22}^*=-M(B_{11}^*M+B_{21}^*).$$

The latter relation gives, for M, the quadratic equation

$$\Big(\frac{B_0(B_1C_1-A_1^2)}{m_0m_1a_{11}^{(1)}}+\frac{B_1(B_0C_0-A_0^2)}{m_0m_1a_{11}^{(0)}}\Big)M^2-$$

$$-2\Big(\frac{A_0(B_1C_1-A_1^2)}{m_0m_1a_{11}^{(1)}}+\frac{A_1(B_0C_0-A_0^2)}{m_0m_1a_{11}^{(0)}}\Big)M+$$
$$+\frac{C_0(B_1C_1-A_1^2)}{m_0m_1a_{11}^{(1)}}+\frac{C_1(B_0C_0-A_0^2)}{m_0m_1a_{11}^{(0)}}=0,$$

whose discriminant is equal to $-T^2$, where

$$T^2=\frac{(B_0C_0-A_0^2)(B_1C_1-A_1^2)}{m_0m_1a_{11}^{(0)}a_{11}^{(1)}}\Delta+$$
$$+\Big(\frac{\varkappa_N^{(1)}(B_0C_0-A_0^2)}{m_0a_{11}^{(0)}}-\frac{\varkappa_N^{(0)}(B_1C_1-A_1^2)}{m_1a_{11}^{(1)}}\Big)^2>0.$$

Therefore the equation has complex roots. Let us choose M such that

$$M=\frac{\frac{A_0(B_1C_1-A_1^2)}{m_0m_1a_{11}^{(1)}}+\frac{A_1(B_0C_0-A_0^2)}{m_0m_1a_{11}^{(0)}}-iT}{\frac{B_0(B_1C_1-A_1^2)}{m_0m_1a_{11}^{(1)}}+\frac{B_1(B_0C_0-A_0^2)}{m_0m_1a_{11}^{(0)}}}.$$

Introducing the notations $\omega=g_1-Mg_2$ and $\Omega_0=\Omega_2+M\Omega_1$, we obtain, for ω, a singular integral equation of the normal type

$$S\omega(x)+\frac{T}{\pi i}\int_l\frac{\omega(t)\,dt}{t-x}=\Omega_0(x),\quad x\in l. \tag{16}$$

Let us define a character of the end points of the integration segment using the results from [4]. We have the equation

$$\gamma=\frac{1}{2\pi i}\ln\frac{S-T}{S+T}=\frac{1}{2\pi i}\ln\frac{-m_0m_1a_{11}^{(0)}a_{11}^{(1)}(S-T)^2}{(B_0C_0-A_0^2)(B_1C_1-A_1^2)\Delta}=\frac{1}{2}-i\beta,$$
$$\beta=\frac{1}{2\pi}\ln\frac{m_0m_1a_{11}^{(0)}a_{11}^{(1)}(S-T)^2}{(B_0C_0-A_0^2)(B_1C_1-A_1^2)\Delta}.$$

Therefore all end points are nonsingular. The solution of (16) is sought for in the class of unbounded functions at the end points.

A canonical solution of the corresponding Hilbert problem in the class of unbounded solutions will have the form

$$X(z)=\prod_{p=1}^{n}(z-a_p)^{-\gamma}(z-b_p)^{\gamma-1},\quad z=x+iy,$$

where we mean the branch defined by the conditions $z^nX(z)\to1$ as $z\to\infty$.

Since the order of the canonical solution at infinity is equal to $-n$, the index of the class of unbounded solutions is $\varkappa=n$. Hence it follows that equation (16) is always solvable in this class and the solution will be written as

$$\omega(x)=\frac{S}{S^2-T^2}\Omega_0(x)-\frac{TX^+(x)}{\pi i(S^2-T^2)}\int_l\frac{\Omega_0(t)\,dt}{X^+(t)(t-x)}+$$
$$+\frac{T}{S-T}X^+(x)P_{n-1}(x),\quad x\in l, \tag{17}$$

where $X^+(x)$ is the boundary value of the canonical solution $X(z)$ on l from D_0; $P_{n-1}(x)$ is the polynomial of degree $n-1$ with arbitrary complex coefficients, $P_{n-1}(x)=K_0x^{n-1}+K_1x^{n-2}+\cdots+K_{n-1}$.

Obviously, having defined ω, we thereby define the vector $\mathbf{g}$ which will be unbounded near points a_p, b_p, $p=1,2,\dots,n$, and will linearly depend on $2n$ arbitrary real constants $\operatorname{Re}K_j$ and $\operatorname{Im}K_j$, $j=0,1,2,\dots,n-1$,

$$\mathbf{g}=-\frac{1}{\operatorname{Im}M}\begin{pmatrix}\operatorname{Im}(\omega\bar{M})\\ \operatorname{Im}\omega\end{pmatrix}. \tag{18}$$

Let us choose these arbitrary real constants such that the vector $\mathbf{g}$ satisfy condition (9). The latter condition gives, for the unknown constants, a system of $2n$ algebraic equations with the same number of unknowns. This system is always solvable. Indeed, the homogeneous system obtained in the case of the boundary functions $\mathbf{f}=\boldsymbol{\varphi}=\mathbf{F}=\Phi=0$ cannot have nontrivial solutions. Then, as one can easily establish by the uniqueness theorem, the original problem has only the trivial solution. Therefore the nonhomogeneous problem is always solvable uniquely.

We have thus proved

Theorem 1. *If the conditions* a), b), c), d), e) *are fulfilled, then the stress problem with the boundary-contact conditions* (3) *and* (4) *always has the unique solution to within an additive constant. The solution is given by formula* (8)*, where the vector* $\mathbf{h}$ *is defined by equality* (13) *and the vector* $\mathbf{g}$ *by equalities* (18) *and* (17)*.*

Taking into account the behaviour of Cauchy-type integrals near the ends of integration lines one can easily obtain the asymptotics of stress components at the vertices of slits in the case of concrete boundary data.

Mixed Problem. Let, the vectors $\mathbf{f},\boldsymbol{\varphi},\mathbf{F},\Phi$ satisfy the conditions a),b),d) of the stress problem and the conditions

c′) $\mathbf{f}',\boldsymbol{\varphi},\mathbf{F}$ belong to the class H^*, and Φ belongs to the Hölder class and has a derivative from the class H^*;

e′) the resultant vector of force applied to the lower slit edges is given and there are no stress and rotation at infinity.

Since the boundary values of $\tau_{xy}^{(0)}$ and $\sigma_y^{(0)}$ are given on the upper slit edges, it is obvious that the resultant vector of external force applied to the real axis will also be known in this problem and represantable by formula (11).

The displacement vector $(u^{(j)}, v^{(j)})$ in the domain D_j, $j = 0, 1$, will again be sought for in form (8), where the unknown vectors $\mathbf{g}$ and $\mathbf{h}$ satisfy conditons (9) and (10). It is obvious that the stress components $\tau_{xy}^{(j)}$ and $\sigma_y^{(j)}$ are represented by formula (12).

Like in the stress problem, the vector $(u^{(j)}, v^{(j)})$ here also satisfies the first of conditions (3), whereas the vector $\mathbf{h}$ will be known on L by virtue of equality (13) and conditions (3).

Therefore it remains for us to define the vectors $\mathbf{g}$ and $\mathbf{h}$ on l. To this end let us calculate the boundary value of the vector $(\tau_{xy}^{(0)}, \sigma_y^{(0)})$ on l except perhaps points a_p, b_p, $p = 1, 2, \dots, n$. We shall have

$$\begin{pmatrix} \tau_{xy}^{(0)} \\ \sigma_y^{(0)} \end{pmatrix}^{+} = -\frac{1}{\Delta}\mathbf{A}^*\mathbf{g}(x) + \frac{\mathbf{B}^*}{\Delta}\Big(\frac{1}{\pi}\int_L \frac{\mathbf{f}'(t)dt}{t-x} - \frac{1}{\pi}\int_l \frac{\mathbf{g}(t)dt}{t-x}\Big) - M^*\mathbf{h}(x) - \\ -\frac{\mathbf{D}^*}{\pi\Delta}\int_{-\infty}^{+\infty}\frac{\mathbf{h}(t)\,dt}{t-x} + \frac{1}{\pi}\mathrm{Im}\frac{\sum_{k=1}^{2}\mathbf{N}^{(0)}(k)}{x+i}\mathbf{X}^{(0)}\mathbf{P}, \quad x \in l, \tag{19}$$

where

$$\mathbf{M}^* = \frac{1}{\Delta}\Big\{\Big(\frac{B_0C_0 - A_0^2}{m_0 a_{11}^{(0)}} + \varkappa_n^{(1)}\varkappa_N^{(0)}\Big)\mathbf{E} + \\ + \frac{1}{m_0 m_1}\begin{Vmatrix} B_0C_1 - A_0A_1 & A_1B_0 - A_0B_1 \\ A_1C_0 - A_0C_1 & B_1C_0 - A_0A_1 \end{Vmatrix}\Big\}.$$

Calculating now on l the boundary value of the derivative of the vector $(u^{(1)}, v^{(1)})$ with respect to x we obtain that for every x except perhaps points a_p, b_p, $p = 1, 2, \dots, n$,

$$\Big(\frac{\partial}{\partial x}\begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix}\Big)^{-} = \mathbf{A}^{(1)}g(x) + \mathbf{B}^{(1)}\Big(\frac{1}{\pi}\int_L \frac{\mathbf{f}'(t)\,dt}{t-x} - \frac{1}{\pi}\int_l \frac{\mathbf{g}(t)\,dt}{t-x}\Big) - \\ -\mathbf{C}^{(1)}\mathbf{h}(x) + \frac{\mathbf{D}^{(1)}}{\pi}\int_{-\infty}^{+\infty}\frac{\mathbf{h}(t)\,dt}{t-x} - \\ -\frac{m_1}{\pi(B_1C_1 - A_1^2)}\begin{Vmatrix} C_1 & A_1 \\ A_1 & B_1 \end{Vmatrix}\mathbf{X}^{(1)}\mathbf{P}\frac{x}{x^2+1}. \tag{20}$$

By virtue of the boundary conditions (5) relations (19) and (20) give for the vectors $\mathbf{g}$ and $\mathbf{h}$ a system of singular integral equations

$$\begin{aligned} &\mathbf{A}^{(1)}\mathbf{g}(x) - \mathbf{C}^{(1)}\mathbf{h}(x) - \frac{\mathbf{B}^{(1)}}{\pi}\int_l \frac{\mathbf{g}(t)\,dt}{t-x} + \\ &\qquad + \frac{\mathbf{D}^{(1)}}{\pi}\int_l \frac{\mathbf{h}(t)\,dt}{t-x} = \mathbf{\Omega}^{(1)}(x), \\ &\frac{\mathbf{A}^*}{\Delta}\mathbf{g}(x) + \mathbf{M}^*\mathbf{h}(x) + \frac{\mathbf{B}^*}{\pi\Delta}\int_l \frac{\mathbf{g}(t)\,dt}{t-x} + \\ &\qquad + \frac{\mathbf{D}^*}{\pi\Delta}\int_l \frac{\mathbf{h}(t)\,dt}{t-x}\Big) = \mathbf{\Omega}^{(2)}(x), \quad x \in l, \end{aligned} \tag{21}$$

where

$$\begin{aligned} \mathbf{\Omega}^{(1)}(x) &= \Phi'(x) - \frac{\mathbf{B}^{(1)}}{\pi}\int_L \frac{\mathbf{f}'(t)\,dt}{t-x} - \frac{\mathbf{D}^{(1)}}{\pi}\int_L \frac{\mathbf{h}(t)\,dt}{t-x} + \\ &\quad + \frac{m_1}{\pi(B_1C_1 - A_1^2)}\begin{Vmatrix} C_1 & A_1 \\ A_1 & B_1 \end{Vmatrix}\mathbf{X}^{(1)}\mathbf{P}\frac{x}{x^2+1}, \\ \mathbf{\Omega}^{(2)}(x) &= \mathbf{F}(x) + \frac{\mathbf{B}^*}{\pi\Delta}\int_L \frac{\mathbf{f}'(t)\,dt}{t-x} - \frac{\mathbf{D}^*}{\pi\Delta}\int_L \frac{\mathbf{h}(t)\,dt}{t-x} + \\ &\quad + \frac{1}{\pi}\mathrm{Im}\frac{\sum_{k=1}^{2}\mathbf{N}^{(0)}(k)}{x+i}\mathbf{X}^{(0)}\mathbf{P}. \end{aligned}$$

Note that due to the properties of the boundary data the vectors $\mathbf{\Omega}^{(1)}$ and $\mathbf{\Omega}^{(2)}$ will belong to the class H^* on l.

If we introduce the partitioned matrices

$$\mathbf{A} = \begin{Vmatrix} \mathbf{A}^{(1)} & -\mathbf{C}^{(1)} \\ \Delta^{-1}\mathbf{A}^* & \mathbf{M}^* \end{Vmatrix}, \quad \mathbf{B} = \begin{Vmatrix} -\mathbf{B}^{(1)} & \mathbf{D}^{(1)} \\ \Delta^{-1}\mathbf{B}^* & \Delta^{-1}\mathbf{D}^* \end{Vmatrix}$$

and denote

$$\boldsymbol{\omega} = (g_1, g_1, h_1, h_2), \quad \widetilde{\mathbf{\Omega}} = (\mathbf{\Omega}^{(1)}, \mathbf{\Omega}^{(2)}),$$

then system (21) takes the form

$$\mathbf{A}\boldsymbol{\omega}(x) + \frac{\mathbf{B}}{\pi}\int_l \frac{\boldsymbol{\omega}(t)\,dt}{t-x} = \widetilde{\mathbf{\Omega}}(x), \quad x \in l. \tag{22}$$

It is obvious that system (22) is a characteristic system of singular integral equations for the real vector $\boldsymbol{\omega}$ with its right-hand side from the class H^*.

The condition

$$\det(\mathbf{A} \pm i\mathbf{B}) = \frac{B_0C_0 - A_0^2}{m_0 a_{11}^{(0)}\Delta} > 0$$

implies that system (22) is of the normal type.

Thus the theory of systems of singular integral equations in the case of open arcs [6] can be used for system (22).

According to this theory it is required to define the roots of the equation

$$\det\{\mathbf{G}^{-1}(t+0)\mathbf{G}(t-0) - \lambda\widetilde{\mathbf{E}}\} = 0 \tag{23}$$

at end points a_p, b_p, $p = 1, 2, \ldots, n$, where $\mathbf{G} = (\mathbf{A} + i\mathbf{B})^{-1}(\mathbf{A} - i\mathbf{B})$ and $\widetilde{\mathbf{E}}$ denotes the unit matrix of order 4.

After long and cumbersome calculations we ascertain that equation (23) has the same form at all points a_p, b_p, $p = 1, 2, \ldots, n$,

$$\lambda^4 + (b-a)\lambda^3 + 2(b+a-1)\lambda^2 + (b-a)\lambda + 1 = 0, \tag{24}$$

where

$$a = \frac{4}{\Delta}\big(1 + \omega_0 a_{11}^{(0)} \varkappa_N^{(1)}\big)^2 \frac{B_0C_0 - A_0^2}{m_0 a_{11}^{(0)}} > 0, \tag{25}$$

$$b = \frac{4(B_1C_1 - A_1^2)a_{11}^{(0)}}{\Delta m_0 m_1^{(0)}} > 0. \tag{26}$$

It can be easily verified that equation (24) reduces to the following equation:

$$\big(\lambda + \frac{1}{\lambda}\big)^2 + (b-a)\big(\lambda + \frac{1}{\lambda}\big) + 2(b+a-2) = 0. \tag{27}$$

Let us investigate the roots of this equation. Note that the inequality

$$a - b - 4 = -\big[1 - \omega_0^2(B_0C_0 - A_0^2)\big]b - \big[1 - \omega_1^2(B_1C_1 - A_1^2)\big]b - \\ - \frac{4}{\Delta}\,\frac{B_0C_1 + B_1C_0 - 2A_0A_1}{m_0m_1} < 0$$

yields $a < b + 4$.

Consider all possible cases: $a = b$, $a < b$ and $a > b$. In the first case equation (27) has only complex roots; in the second case either it has only complex roots if $b \le 4$ or it has no positive roots if $b > 4$; and, finally, in the third case it has no positive roots if $a \le 4$ and has only complex roots if $a > 4$.

Thus in all three cases the equation has not only positive roots. Therefore all end points a_p, b_p, $p = 1, 2, \ldots, n$, are nonsingular.

A solution of system (22) is to be sought for in the class functions unbounded at end points a_p, b_p, $p = 1, 2, \ldots, n$. Whenever equation (27) has simple complex roots, one can easily construct the solution of system (22), having first constructed the matrix of canonical solutions of the corresponding homogeneous Hilbert problem in the class of unbounded functions.

Indeed, introducing a piecewise holomorphic vector

$$\mathbf{W}(z) = \frac{1}{2\pi i}\int_l \frac{\boldsymbol{\omega}(t)\,dt}{t-z}, \quad z = x + iy,$$

system (22) becomes equivalent in a certain sense to the nonhomogeneous Hilbert problem

$$\mathbf{W}^{+}(z)=\mathbf{G}\mathbf{W}^{-}(x)+\mathbf{R}(x),\quad x\in l, \tag{28}$$

where $\mathbf{R}(x)=(\mathbf{A}+i\mathbf{B})^{-1}\widetilde{\mathbf{\Omega}}(x)$.

Since we are considering the case when equation (27) has only simple complex roots λ_1, λ_2, λ_3, λ_4, it is always possible to construct a nonsingular matrix $\mathbf{S}$ such that the equality $\mathbf{S}^{-1}\mathbf{G}\mathbf{S}=\mathbf{\Lambda}$ be fulfilled, where $\mathbf{\Lambda}$ is a diagonal matrix with the elements λ_1 , λ_2, λ_3, λ_4 on the main diagonal

$$\mathbf{\Lambda}=\operatorname{diag}\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}.$$

Using the constructed matrix $\mathbf{S}$ we rewrite the Hilbert problem (28) as

$$\Psi^{+}(z)=\mathbf{\Lambda}\Psi^{-}(x)+\mathbf{S}^{-1}\mathbf{R}(x),\quad x\in l, \tag{29}$$

where $\Psi(z)=\mathbf{S}^{-1}\mathbf{W}(z)$.

Since $\mathbf{\Lambda}$ is a diagonal matrix, we conclude that in the class of functions unbounded at end points a_p, b_p, $p=1,2,\dots,n$, the matrix of canonical solutions corresponding to the homogeneous Hilbert problem (29) has the form

$$\mathbf{X}(z)=\operatorname{diag}\{X_1(z),X_2(z),X_3(z),X_4(z)\}, \tag{30}$$

where

$$X_\sigma(z)=\prod_{p=1}^{n}(z-a_p)^{-\gamma_\sigma}(z-b_p)^{\gamma_\sigma-1},\quad \gamma_\sigma=\frac{1}{2\pi i}\ln\lambda_\sigma,\quad \sigma=1,2,3,4. \tag{31}$$

By $X_\sigma(z)$, $\sigma=1,2,3,4$, we mean the branch defined by the condition

$$\lim_{z\to\infty}\{z^n X_\sigma(z)\}=1.$$

Since we seek for an unbounded solution at end points a_p, b_p, $p=1,2,\dots,n$, the numbers γ_σ, $\sigma=1,2,3,4$, have to be chosen so that $0<\operatorname{Re}\gamma_\sigma<1$, $\sigma=1,2,3,4$.

In the particular case when the upper and lower half-planes are filled with materials for which the conditions

$$a_{11}^{(0)}=a_{11}^{(1)},\quad B_0=B_1,\quad C_0=C_1,\quad A_0=A_1,\quad \omega_0=\omega_1$$

are satisfied, equation (24) takes the form

$$\lambda^4+2\frac{1+\omega_0^2(B_0C_0-A_0^2)}{1-\omega_0^2(B_0C_0-A_0^2)}\lambda^2+1=0,$$

i.e.,

$$\left(\lambda^2+\frac{1+\omega_0\sqrt{B_0C_0-A_0^2}}{1-\omega_0\sqrt{B_0C_0-A_0^2}}\right)\left(\lambda^2+\frac{1-\omega_0\sqrt{B_0C_0-A_0^2}}{1+\omega_0\sqrt{B_0C_0-A_0^2}}\right)=0.$$

Hence it is clear that in this particular case equation (24) has purely imaginary roots

$$\lambda_1=i\sqrt{\varkappa_0},\quad \lambda_2=-i\sqrt{\varkappa_0},\quad \lambda_3=\frac{i}{\sqrt{\varkappa_0}},\quad \lambda_4=-\frac{i}{\sqrt{\varkappa_0}},$$

where

$$\varkappa_0=\frac{1+\omega_0\sqrt{B_0C_0-A_0^2}}{1-\omega_0\sqrt{B_0C_0-A_0^2}}$$

and the numbers γ_σ, $\sigma=1,2,3,4$, have the form

$$\gamma_1=\frac{1}{4}-i\beta_0,\quad \gamma_2=\frac{3}{4}-i\beta_0,\quad \gamma_3=\bar{\gamma}_1,\quad \gamma_4=\bar{\gamma}_2,\quad \beta_0=\frac{1}{4\pi}\ln\varkappa_0.$$

By the general theory of systems of singular integral equations [6], from (30) and (31) it follows that the partial indices from the class of unbounded at end points solutions of the homogeneous Hilbert problem satisfy the equations $\varkappa_1=\varkappa_2=\varkappa_3=\varkappa_4=n$ and the total index $\varkappa=4n$.

Therefore the nonhomogeneous Hilbert problem (29) will always be solvable in the class of solutions unbounded at end points, and the solution will depend on $4n$ arbitrary constants

$$\Psi(z)=\frac{\mathbf{X}(z)}{2\pi i}\int_l\frac{[\mathbf{X}^+(t)]^{-1}\mathbf{S}^{-1}\mathbf{R}(t)\,dt}{t-z}+\mathbf{X}(z)\mathbf{P}(z), \tag{32}$$

where $\mathbf{P}(z)=(P_1(z),P_2(z),P_3(z),P_4(z))$ and $P_\sigma(z)$, $\sigma=1,2,3,4$, is a polynomial of degree $n-1$ with arbitrary real coefficients.

After finding the vector Ψ, the vector $\mathbf{W}$ is defined by the equality $\mathbf{W}=\mathbf{S}\Psi$ and, finally, the solution of system (22), unbounded at end points a_p, b_p, $p=1,2,\dots,n$, is given by the formula $\boldsymbol{\omega}=\mathbf{W}^+-\mathbf{W}^-$.

Thus we have found the solution of system (22) which will depend on $4n$ arbitrary real constants.

It is obvious that the vectors $\mathbf{g}$ and $\mathbf{h}$ and, accordingly, the displacement vectors will depend linearly on the same $4n$ constants.

To define these constants we have the following conditions. In the first place, the found vectors $\mathbf{g}$ and $\mathbf{h}$ should satisfy conditions (9) and (10), which gives $2n+2$ linear equations. On the other hand, from the above arguments it is clear that on the lower slit edges the displacement vector will satisfy the conditions

$$\begin{pmatrix}u^{(1)}\\ v^{(1)}\end{pmatrix}^-=\Phi(x)+\boldsymbol{\eta}_p\quad\text{on }\ l_p,\ p=1,2,\dots,n,$$

where $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \dots, \boldsymbol{\eta}_p$ are arbitrary real constant vectors.

We obtain more $2n-2$ linear algebraic equations provided that $\boldsymbol{\eta}_1 = \boldsymbol{\eta}_2 = \cdots \boldsymbol{\eta}_n = \boldsymbol{\eta}$, where $\boldsymbol{\eta}$ is an arbitrary real constant vector.

Thus we have the system of $4n$ algebraic equations for defining $4n$ unknowns and, by virtue of the uniqueness theorem, we readily conclude that this system is always solvable uniquely.

Finally, we observe that the vector

$$\begin{pmatrix} u^{(j)} \\ v^{(j)} \end{pmatrix} - \boldsymbol{\eta}, \quad j = 0, 1,$$

is the solution of the mixed problem.

Thus we have proved

Theorem 2. *If conditions* a), b), c′), d), e′) *are fulfilled, then under the boundary-contact conditions the mixed problem* (3) *and* (5) *has the unique solution which is represented by formula* (8)*, where the vector* $\mathbf{h}$ *is defined on* L *by equality* (13) *and the vectors* $\mathbf{g}$ *and* $\mathbf{h}$ *are defined on* l *by the solution of system* (22)*.*

References

1. M.O. Basheleishvili, Solution of plane boundary value problems of statics of an anisotropic elastic body. (Russian) *Akad. Nauk. Gruz. SSR. Trudy Vichislit. Tsentra, v.* III, 1963, 93-139.

2. P.I. Plakhotnyi, Reciprocal pressure of two different anisotropic half-planes with rectilinear slits on the junction line. (Russian) *Prikladnaya mekhanika,* **6**(1970), No. 1, 62-68.

3. S.G. Lekhnitskii, Anisotropic plates. (Russian) *OGIZ Gostekhizdat, Moscow-Leningrad,* 1947.

4. N.I. Muskhelishvili, Singular integral equations. (Translation from the Russian) *P.Noordhoff (Ltd.), Groningen,* 1953.

5. M.O. Basheleishvili, Solution of the basic contact problems for a nonhomogeneous anisotropic plane. (Russian) *Some Problems of Elasticity Theory.* (Russian) *Proceedings of I.N. Vekua Institute of Applied Mathematics* v. 39, 13-26, *Tbilisi University Press, Tbilisi*, 1990.

6. N.P. Vekua, Systems of singular integral equations. (Translation from the Russian) *P.Noordhoff (Ltd.), Groningen,* 1967.

7. L.A. Galin, Contact problems of the theory of elasticity and viscoelasticity. (Russian) *"Nauka", Moscow,* 1980.

(Received 14.12.1992)

Author's address:
I.Vekua Institute of Applied Mathematics
of Tbilisi State University
2 University St., 380043 Tbilisi
Republic of Georgia