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LOCAL GROWTH OF WEIERSTRASS σ-FUNCTION AND WHITTAKER-TYPE DERIVATIVE SAMPLING

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Dedicated to N. N. Leonenko on the occasion of his 50th birthday

Abstract. Two explicit guard functions $K_j = K_j(\delta_z)$, $j = 1, 2$, are obtained, which depend on the distance δ_z between z and the nearest point of the integer lattice in the complex plane, such that $\delta_z K_1(\delta_z) \le |\sigma(z)|e^{-\pi|z|^2/2} \le \delta_z K_2(\delta_z)$, $z \in \mathbb{C}$, where $\sigma(z)$ stands for the Weierstraß σ-function. This result is used to improve the circular truncation error upper bound in the q-th order Whittaker-type derivative sampling for the Leont'ev functions space $[2, \frac{\pi q}{2})$, $q \ge 1$.

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1. Introduction

The Weierstraß σ-function described as an infinite product is given by

$$\sigma(z) = z \prod_{(m,n)\in\mathbb{Z}^2}{}' \left(1 - \frac{z}{m+ni}\right) \exp\left\{\frac{z}{m+ni} + \frac{z^2}{2(m+ni)^2}\right\},$$

where the dashed product means that the factor with $m = n = 0$ is omitted. The local growth estimation of $\sigma(z)$, interesting for different purposes, has quite a long history. The first result of this kind known by the author is given in [3], where $\ln M_\sigma(r) \sim \pi r^2/2$, $r \to \infty$, is proved ($M_\sigma(r)$ stands for the maximum modulus of σ on the circle $|z| = r$), see also [7, Chapter 4, §1, Problem 49], where this result is quoted from the book [3]. After that Hayman proved that there exist absolute constants $\mathbf{K}_1, \mathbf{K}_2$, $\mathbf{K}_1 < \mathbf{K}_2$, for which

$$\mathbf{K}_1 \le \frac{|\sigma(z)|}{\operatorname{dist}(z, \mathbb{Z}^2)}\, e^{-\pi|z|^2/2} \le \mathbf{K}_2, \qquad z \in \mathbb{C}. \tag{1}$$

In his article [1] no specific comments were given on the nature of $\mathbf{K}_j$'s. (Here we have to point out that Hayman's proof is not correct; he wrongly deduced that the type of $\sigma(z)$ is $\pi/4$ instead of the true value $\pi/2$, [1].) Seip confirms (1) on the lattice $\Lambda_\alpha = \sqrt{\frac{\pi}{\alpha}}\mathbb{Z}^2$, $\alpha > 0$, but without any closer specification of $\mathbf{K}_j$'s, [8] (these estimates of $\sigma(z)$ were needed as convergence tools).

In this short note we obtain the values for these uniform constants ($\mathbf{K}_1 \approx 0.266$ and $\mathbf{K}_2 = 1$) as a corollary of our main result. Our principal result is to

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establish two positive guard functions $K_j = K_j(\delta_z)$, $j = 1, 2$, depending just on $\delta_z = \operatorname{dist}(z, \mathbb{Z}^2)$ such that (1) holds below; the proposed numerical bounds (see the Proof of the Corollary 1), are the minimal and maximal values of these guard functions respectively (therefore $\mathbf{K}_j$ cannot be improved in our setting).

In the third section of the article we consider the truncation error upper bound appearing in the Whittaker-type derivative sampling restoration formula, i.e. the sampling restoration formula which involves not just the sampled values of the function f, but the sampled values of $f^{(j)}$, $j = \overline{0, q-1}$, i.e., of the first q derivatives of the functions sampled at points of the lattice $\mathbb{Z}^2$. This type of sampling reconstruction holds for the Leont'ev type functions space $[2, \frac{\pi q}{2})$ [1], [4]. Our second main goal is to improve the truncation error upper bound in [6], Theorem 1, with the aid of inequalities (2).

2. Local Growth Behaviour of $\sigma(z)$-Function

Along with the notation already introduced the following symbols will also be used: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ denote the sets of natural, integer and complex numbers, respectively; the symbol $\operatorname{Cl}\{A\}$ denotes the closure of the set A, $t^\star$ is the complex conjugate of $t \in \mathbb{C}$, the circle $\Gamma_r = \{\zeta | \, |\zeta| = r\}$ contains no point of $\mathbb{Z}^2$.

Theorem 1. *For all $z \in \mathbb{C}$ we have*

$$\delta_z K_1(\delta_z) \le |\sigma(z)| \exp\left\{-\frac{\pi}{2}|z|^2\right\} \le \delta_z K_2(\delta_z), \tag{2}$$

where

$$K_1(\delta_z) = \left(1 - \frac{\pi^4 \delta_z^4}{90}\right)\left(1 - A_{22}\delta_z^4\right)^2 \exp\left\{-\frac{\pi}{2}\delta_z^2\right\}, \tag{3}$$

$$K_2(\delta_z) = \exp\left\{\left(\frac{\pi^4}{90} + A_{22}\right)\delta_z^4 - \frac{\pi}{2}\delta_z^2\right\}, \tag{4}$$

and $A_{22} := \sum_{m,n\in\mathbb{N}} (m^2 + n^2)^{-2}$.

Proof. Let $P_{\frac{1}{2}} = (-\frac{1}{2}, \frac{1}{2})^2$ be the period cell of the Weierstraß σ-function. First we remark that for all $z \in \operatorname{Cl}\{P_{\frac{1}{2}}\}$,

$$\delta_z (1 - A_{22}\delta_z^4)^2 \left(1 - \frac{\pi^4 \delta_z^4}{90}\right) \le |\sigma(z)| \le \delta_z \exp\left\{\left(\frac{\pi^4}{90} + A_{22}\right)\delta_z^4\right\}. \tag{5}$$

Indeed, if $z = x + iy \in \operatorname{Cl}\{P_{\frac{1}{2}}\}$, then we conclude that $\delta_z = \operatorname{dist}(z, \mathbb{Z}^2) = \operatorname{dist}(z, 0) = |z|$. Put

$$a_{m,n} := \left(1 - \frac{z}{m + ni}\right) \exp\left\{\frac{z}{m + ni} + \frac{z^2}{2(m + ni)^2}\right\}.$$

[1]The infrequently used notation $[\rho, \psi)$, $[\rho, \psi]$ stands for the functions space consisting of all entire functions of order less then or equal to ρ, and $[\rho, \psi)$ denotes the case when the function of order ρ has a type less then ψ, while $[\rho, \psi]$ is used in the case when the function of order ρ possesses a type less then or equal to ψ, [4].

Consequently, we directly get

$$
\begin{aligned}
|\sigma(z)|^2 &= |z|^2 \prod_{n\in\mathbb{N}} |a_{n,0}a_{-n,0}a_{0,n}a_{0,-n}|^2 \prod_{m,n\in\mathbb{N}} |a_{m,n}a_{m,-n}a_{-m,n}a_{-m,-n}|^2 \\
&= |z|^2 \prod_{n\in\mathbb{N}} \left| \left(1 - \frac{z^2}{n^2}\right) \left(1 + \frac{z^2}{n^2}\right) \right|^2 \\
&\times \prod_{m,n\in\mathbb{N}} \left| \left(1 - \left(\frac{z}{m+ni}\right)^2\right) \left(1 - \left(\frac{z}{m-ni}\right)^2\right) \right. \\
&\times \left. \exp\left\{ z^2 \left(\frac{1}{(m+ni)^2} + \frac{1}{(m-ni)^2} \right) \right\} \right|^2 \qquad (5a) \\
&\le |z|^2 \prod_{n\in\mathbb{N}} \left(1 + \frac{|z|^4}{n^4}\right)^2 \qquad (5b) \\
&\times \prod_{m,n\in\mathbb{N}} \left(1 - 2\frac{(x^2-y^2)(m^2-n^2) + 4mnxy}{(m^2+n^2)^2} + \frac{|z|^4}{(m^2+n^2)^2}\right) \qquad (5c) \\
&\times \left(1 - 2\frac{(x^2-y^2)(m^2-n^2) - 4mnxy}{(m^2+n^2)^2} + \frac{|z|^4}{(m^2+n^2)^2}\right) \qquad (5d) \\
&\times \exp\left\{ 4\frac{(x^2-y^2)(m^2-n^2)}{(m^2+n^2)^2} \right\}.
\end{aligned}
$$

Now, we use $1+t \le e^t$ to majorize the factors in displays $(5b, c, d)$. So, straightforward calculation results in

$$
|\sigma(z)|^2 \le |z|^2 \exp\left\{ 2\left(\frac{\pi^4}{90} + \sum_{m,n\in\mathbb{N}} \frac{1}{(m^2+n^2)^2} \right) |z|^4 \right\}, \qquad (6)
$$

which is the asserted upper bound in (5).

The lower bound derivation will be realized in a somewhat different way. At first an auxiliary inequality is established. Namely, let $a \in \mathbb{C}$, $|a| < 1$. Then there holds

$$
|(1-a)e^a| \ge 1 - |a|^2. \qquad (7)
$$

Indeed, after fixing $a = |a|e^{i\phi}$, $|a|$, we get by direct calculation

$$
\begin{aligned}
h(\phi) &:= |(1-a)e^a| = e^{|a|\cos\phi}\sqrt{1 - 2|a|\cos\phi + |a|^2} \\
&\ge \min_{\phi\in[0,2\pi)} h(\phi) = h(0) = (1-|a|)e^{|a|}.
\end{aligned}
$$

Finally, by $e^t \ge 1+t$ we deduce estimate (7).

To continue, assume $\lambda_n \in (0,1), n \in \mathbb{N}$. Then if $\sum_{n\in\mathbb{N}} \lambda_n$ converges, there holds

$$
\prod_{n\in\mathbb{N}} (1-\lambda_n) \ge 1 - \sum_{n\in\mathbb{N}} \lambda_n. \qquad (8)
$$

This result is a generalization of the inequality due to Weierstraß. (In fact, (8) is a straightforward consequence of the inequality concerning the finite product/sum case, cf. (1) in [5, **3.2.37.**, p. 207]).

Now we are ready to establish the lower bound for $|\sigma(z)|, z \in \mathrm{Cl}\{P_{\frac{1}{2}}\}$. We have

$$\begin{aligned}
|\sigma(z)| = |z| & \left| \prod_{n\in\mathbb{N}} \left(1 - \frac{z^4}{n^4}\right) \right| \\
& \times \left| \prod_{m,n\in\mathbb{N}} \left(1 - \frac{z^2}{(m+ni)^2}\right) e^{\frac{z^2}{(m+ni)^2}} \left(1 - \frac{z^2}{(m-ni)^2}\right) e^{\frac{z^2}{(m-ni)^2}} \right| \qquad (9) \\
& \geq |z| \prod_{n\in\mathbb{N}} \left(1 - \frac{|z|^4}{n^4}\right) \prod_{m,n\in\mathbb{N}} \left(1 - \frac{|z|^4}{(m^2+n^2)^2}\right)^2 \qquad (10) \\
& \geq |z| \left(1 - |z|^4 \sum_{n\in\mathbb{N}} \frac{1}{n^4}\right) \left(1 - |z|^4 \sum_{m,n\in\mathbb{N}} \frac{1}{(m^2+n^2)^2}\right)^2 \\
& = |z| \left(1 - \frac{\pi^4}{90}\delta_z^4\right) \left(1 - A_{22}\delta_z^4\right)^2 . \qquad (11)
\end{aligned}$$

Indeed, since for $a = z^2(m \pm in)^{-2}$ we have $|a| \leq \frac{1}{4}$ for all $z \in \mathrm{Cl}\{P_{\frac{1}{2}}\}$, $m, n \in \mathbb{N}$, this allows us to use estimate (7) in the double-indexed product in (9), and then, taking into account that the factors $\lambda_n^{(1)} = |z|^4 n^{-4}$, $\lambda_n^{(2)} = |z|^4(m^2+n^2)^{-2}$ belong to the interval $(0,1)$ in display (10), with the help of the Weierstraß-type inequality (8) we easily deduce (11).

Finally, combining estimates (6) and (11), we finish the derivation of (5).

Let $\tau_{\mathbf{k}} : \mathbb{C} \mapsto \mathrm{Cl}\{P_{\frac{1}{2}}\}$ be the translation which defines a unique $\mathbf{k} = (k_u, k_v) \in \mathbb{Z}^2$ and a unique $w = (u, v) \in \mathrm{Cl}\{P_{\frac{1}{2}}\}$ such that

$$z = \mathbf{k} + w = k_u + ik_v + u + iv.$$

The quasi-periodicity property of the Weierstraß σ-function implies that

$$\sigma(z) = (-1)^{k_u+k_v+k_uk_v}\sigma(w)\exp\{\pi w\mathbf{k}^\star + \pi|\mathbf{k}|^2/2\}.$$

Since $\mathrm{dist}(z, \mathbb{Z}^2) = \mathrm{dist}(w, 0) = \delta_z$, we get

$$\begin{aligned}
|\sigma(z)| &= |\sigma(w)||\exp\{\pi w\mathbf{k}^\star + \pi|\mathbf{k}|^2/2\}| \\
&\leq \delta_z \exp\left\{\left(\frac{\pi^4}{90} + A_{22}\right)\delta_z^4 + \frac{\pi}{2}(|\mathbf{k}|^2 + 2k_uu + 2k_vv + |w|^2)\right\}\exp\left\{-\frac{\pi}{2}|w|^2\right\} \\
&= \delta_z \exp\left\{\left(\frac{\pi^4}{90} + A_{22}\right)\delta_z^4 - \frac{\pi}{2}\delta_z^2\right\}\exp\left\{\frac{\pi}{2}|z|^2\right\}.
\end{aligned}$$

This finishes the proof of the upper bound assertion in (2). Finally it remains to derive the lower bound in (2). For this we can argue in a similar way so that by the left-hand estimate in (5) we deduce

$$|\sigma(z)| \;\; = \;\; |\sigma(w)||\exp\{\pi w\mathbf{k}^\star + \pi|\mathbf{k}|^2/2\}|$$

$$\geq \quad \delta_z \left(1 - \frac{\pi^4 \delta_z^4}{90}\right) \left(1 - A_{22}\delta_z^4\right)^2 \exp\left\{-\frac{\pi}{2}\delta_z^2\right\} \exp\left\{\frac{\pi}{2}|z|^2\right\},$$

which is the asserted lower bound. □

Remark 1. The *Mathematica* 4.0 gives $A_{22} = \sum_{m,n\in\mathbb{N}}(m^2+n^2)^{-2} \approx 0.42437977$. So we can rearrange (3) and (4) with the approximate value of A_{22}.

Corollary 1.1. *The following estimates hold:*

$$\delta_z \left(1 - \tfrac{\pi^4}{360}\right) \left(1 - \tfrac{A_{22}}{4}\right)^2 \exp\left\{-\frac{\pi}{4}\right\} \leq |\sigma(z)| \exp\left\{-\frac{\pi}{2}|z|^2\right\} \leq \delta_z, \qquad z \in \mathbb{C}.$$

Proof. We find $\min K_1(\delta_z)$ and $\max K_2(\delta_z)$ by (3), (4) as $0 \leq \delta_z \leq 1/\sqrt{2}$. So

$$\mathbf{K}_1 \Leftarrow \min_{[0,1/\sqrt{2}]} K_1(\delta_z) = K_1(1/\sqrt{2}) = \left(1 - \tfrac{\pi^4}{360}\right)\left(1 - \tfrac{A_{22}}{4}\right)^2 e^{-\frac{\pi}{4}} \approx 0,26574548,$$

$$\mathbf{K}_2 \Leftarrow \max_{[0,1/\sqrt{2}]} K_2(\delta_z) = K_2(0) = 1$$

are the uniform Hayman's constants. □

Remark 2. The sharpness of inequalities (2) is an open question. Namely, starting with (9) and (5a), by the Euler's infinite product representation of $\sin \pi\delta_z$ and $\sinh \pi\delta_z$ we get the lower and upper guard functions for $|\sigma(z)|$ as follows:

$$\widetilde{K}_1(\delta_z) = \left(\frac{\sin \pi\delta_z}{\pi\delta_z}\right)^2 (1 - A_{22}\delta_z^4)^2 \exp\left\{-\frac{\pi}{4}\delta_z^2\right\},$$

$$\widetilde{K}_2(\delta_z) = \left(\frac{\sinh \pi\delta_z}{\pi\delta_z}\right)^2 \exp\left\{A_{22}\delta_z^4 - \frac{\pi}{4}\delta_z^2\right\}.$$

Since $\widetilde{K}_1(\delta_z)$ possesses lower minimum than $K_1(\delta_z)$, and $\widetilde{K}_2(\delta_z)$ has larger maximum than $K_2(\delta_z)$ we propose the use of (3) and (4) respectively.

3. Whittaker Type Derivative Sampling

In this part of the article we consider the influence of bounds (2) on the convergence rate in the Whittaker-type q-th order derivative, uniformly spaced sampling restoration formula for the Leont'ev function space $[2, \frac{\pi q}{2})$.

For $f \in [2, \frac{\pi q}{2}\vartheta], \vartheta \in [0,1)$ we have

$$f(z) = \sigma^q(z) \sum_{(m,n)\in\mathbb{Z}^2} \sum_{j=0}^{q-1} \sum_{k=0}^{q-1-j} \frac{f^{(q-1-j-k)}(m+ni) R^q_{mnj}}{j!(q-1-j-k)!(z-m-ni)^{k+1}} \tag{12}$$

uniformly on the compact subsets of $\mathbb{C}$; as usual, $f^{(0)}(\cdot) \equiv f(\cdot)$. Here

$$R^q_{mnj} = \lim_{w\to m+ni} \frac{d^j}{dw^j}\left(\frac{w-m-ni}{\sigma(w)}\right)^q;$$

cf. [6], Theorem 1 and Corollary 1 for certain additional details, e.g. truncation error analysis, etc. (In fact, formula (12) for $q = 1$ belongs to Whittaker; the case $q > 1$ appears in [2] as well.)

Let us introduce a subset of $\mathbb{Z}^2$:

$$\mathbf{N}(r) := \{(m,n)|\,|m+ni|<r\}.$$

Under the Whittaker-type derivative sampling series (12) truncated to $\mathbf{N}(r)$ we mean the interpolation formula

$$\mathcal{I}_N(z;f;\sigma;q;r) = \sum_{(m,n)\in\mathbf{N}(r)} \sum_{j=0}^{q-1} \sum_{k=0}^{q-1-j} \frac{\sigma^q(z) f^{(q-1-j-k)}(m+ni) R^q_{mnj}}{j!(q-1-j-k)!(z-m-ni))^{k+1}}; \quad (13)$$

the so-called *circular truncation error* is

$$\epsilon_N(f;z;q;r) := f(z) - \mathcal{I}_N(z;f;\sigma;q;r). \tag{14}$$

Now, we apply estimates (2) to the results in [6], Theorem 1, Corollary 1, where we cannot avoid truncation error bound estimates such that depend on the Hayman constants $\mathbf{K}_1, \mathbf{K}_2$. To prove the principal result in this section, we will need the following evaluation.

Lemma 1. *Consider a circle $\Gamma_r = \{\zeta|\,|\zeta| = r\}$ such that contains no point of $\mathbb{Z}^2$. Then there holds*

$$\delta_\zeta \geq \mathrm{dist}(\Gamma_r, \mathbb{Z}^2) \geq \frac{1-|1-2(r^2-[r^2])|}{4\,r+\sqrt{2}} := H(r), \tag{15}$$

where $[x]$ is the largest integer less than or equal to x.

Proof. Let $A(\mathfrak{a}_1,\mathfrak{a}_2)$ be the closest point in $\mathbb{Z}^2$ to Γ_r and denote by Δ_A the distance between A and the nearest point on Γ_r. Clearly, $\sqrt{\mathfrak{a}_1^2+\mathfrak{a}_2^2} = r \pm \Delta_A$, choosing the sign according to where A lies, outside or inside the circle Γ_r. Thus we can write $\sqrt{\mathfrak{a}_1^2+\mathfrak{a}_2^2} \leq r + \Delta_A \leq r + 1/\sqrt{2}$. Since $\mathfrak{a}_1^2+\mathfrak{a}_2^2$ is the positive integer nearest to r^2, we have

$$\mathfrak{a}_1^2+\mathfrak{a}_2^2 = [r^2] \quad \text{if } A \text{ lies inside of } \Gamma_r \quad \big(= [r^2]+1 \quad \text{if } A \text{ lies outside of } \Gamma_r\big)\,,$$

which gives, that

$$\begin{aligned}\Delta_A = \mathrm{dist}(\Gamma_r,\mathbb{Z}^2) &= \left| r - \sqrt{\mathfrak{a}_1^2+\mathfrak{a}_2^2}\right| = \frac{|r^2-(\mathfrak{a}_1^2+\mathfrak{a}_2^2)|}{r+\sqrt{\mathfrak{a}_1^2+\mathfrak{a}_2^2}} \\ &\geq \frac{\min\{r^2-[r^2], [r^2]+1-r^2\}}{2r+1/\sqrt{2}}.\end{aligned}$$

According to the definition of a modulus this implies (15). □

Theorem 2. *For all $f \in [2, \frac{\pi q}{2}\vartheta]$, $\vartheta \in [0,1)$, and for all*

$$N \geq \left[\frac{|z|}{\sqrt{2(1-\vartheta)}} + \frac{1}{2}\right] + 1, \quad z \in \mathbb{C},$$

we have

$$|\epsilon_N(f;z;q;\sqrt{2}(N+1/2))| \leq \frac{A_f(2N+1)(4N+3)^q e^{-2\pi q(1-\vartheta)N}}{\mathbf{K}_1^q(2(1-\sqrt{1-\vartheta})N+1+\sqrt{1-\vartheta})}\,, \tag{16}$$

where A_f is the absolute constant which characterizes the $[2, \frac{\pi q}{2}\vartheta]$-function f, i.e. it is given by $|f(z)| \le A_f \exp\left\{\frac{\pi q \vartheta}{2}\,|z|^2\right\}$, $z \in \mathbb{C}$. Moreover

$$\lim_{N\to\infty} \mathcal{I}_N(z; f; \sigma; q; r) = f(z)$$

uniformly in $z \in \mathbb{C}$.

Proof. First we repeat the procedure for deriving the truncation error upper bound in [6], Theorem 1 with the integration path $\Gamma_r = \{\zeta |\, |\zeta| = r\}$ chosen according to the definition of the index set $\mathbf{N}(r)$ and assume $z \in \mathrm{int}(\Gamma_r)$. We point out that $|f(z)| \le A_f \exp\left\{\frac{\pi q}{2}\vartheta r^2\right\}$ on the circle Γ_r with $A_f > 0$. Then using (2), the numerical values of the Hayman constants $\mathbf{K}_1, \mathbf{K}_2$ and (15) in Lemma 1 for estimating $|\sigma(\cdot)|$ in this result in [6], we get

$$\begin{aligned}
|\epsilon_N(f; z; q; r)| &\le \frac{|\sigma(z)|^q}{2\pi} \oint_{\Gamma_r} \frac{|f(\zeta)||d\zeta|}{|\sigma(\zeta)|^q|\zeta - z|} \\
&\le \left(\frac{\delta_z K_2(\delta_z)}{\min_{\zeta\in\Gamma_r} \delta_\zeta K_1(\delta_\zeta)}\right)^q \frac{A_f\ r}{r - |z|}\ \exp\left\{\frac{\pi q}{2}(|z|^2 - (1-\vartheta)r^2)\right\} \\
&\le \left(\frac{\delta_z \mathbf{K}_2}{H(r)\mathbf{K}_1}\right)^q \frac{A_f\ r}{r - |z|}\ \exp\left\{\frac{\pi q}{2}(|z|^2 - (1-\vartheta)r^2)\right\} \\
&\le \frac{A_f\ r(4r + \sqrt{2})^q \exp\left\{\frac{\pi q}{2}(|z|^2 - (1-\vartheta)r^2)\right\}}{(r - |z|)\left(\sqrt{2}\mathbf{K}_1(1 - |1 - 2(r^2 - \lfloor r^2 \rfloor)|)\right)^q}.
\end{aligned} \tag{17}$$

It is not difficult to see that the circle $\Gamma_r,\ r = \sqrt{2}(N + 1/2)$, does not contain any integer point from $\mathbb{Z}^2$, being $r^2 \notin \mathbb{N}$. So $\Gamma_{\sqrt{2}(N+1/2)}$ is a suitable integration contour for which (17) holds. Then, substituting $r = \sqrt{2}(N + 1/2)$ into the bound (17), we deduce

$$\begin{aligned}
&|\epsilon_N(f; z; q; \sqrt{2}(N + 1/2))| \\
&\quad\le \frac{A_f(N + 1/2)(4(N + 1/2) + 1)^q}{\mathbf{K}_1^q((N + 1/2 - \sqrt{1-\vartheta}(N - 1/2))}\ e^{\pi q(1-\vartheta)[(N-1/2)^2-(N+1/2)^2]} \\
&\quad= \frac{A_f(2N + 1)(4N + 3)^q \exp\{-2\pi q(1-\vartheta)N\}}{\mathbf{K}_1^q(2(1 - \sqrt{1-\vartheta})N + 1 + \sqrt{1-\vartheta})},
\end{aligned} \tag{18}$$

which is the asserted upper bound (16).

The uniform convergence in $f(z) \approx \mathcal{I}_N(z; f; \sigma; q)$ follows from the truncation error upper bound (16) as $N \to \infty$. Indeed, since the right-hand term in (16) does not depend on z and vanishes with the growth of N, the assertion follows. □

Remark 3. By fixing the values of z we get the mathematical model

$$\frac{A_f(2N + 1)(4N + 3)^q e^{-2\pi q(1-\vartheta)N}}{\mathbf{K}_1^q(2(1 - \sqrt{1-\vartheta})N + 1 + \sqrt{1-\vartheta})} < \epsilon \tag{19}$$

for a pre-assigned approximation error level $\epsilon > 0$ from (16). Then inequality (19) gives an optimal value of N in finding the minimal size of the approximation sum (13). Moreover, the convergence rate in (12) is

$$|\epsilon_N(f;z)| = \mathcal{O}(N^q e^{-2\pi q(1-\vartheta)N})$$

under the assumptions of Theorem 2.

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