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ABOUT ASYMPTOTIC AND OSCILLATION PROPERTIES OF THE DIRICHLET PROBLEM FOR DELAY PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, oscillation and asymptotic properties of solutions of the Dirichlet boundary value problem for hyperbolic and parabolic equations are considered. We demonstrate that introducing an arbitrary constant delay essentially changes the above properties. For instance, the delay equation does not inherit the classical properties of the Dirichlet boundary value problem for the heat equation: the maximum principle is not valid, unbounded solutions appear while all solutions of the classical Dirichlet problem tend to zero at infinity, for "narrow enough zones" all solutions oscillate instead of being positive. We establish that the Dirichlet problem for the wave equation with delay can possess unbounded solutions. We estimate zones of positivity of solutions for hyperbolic equations.

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1. Introduction

In this paper the following equation with memory

$$Lv(\cdot,x)(t) + Tv''_{xx}(\cdot,x)(t) = 0, \quad x \in [0,\omega],\ t \in [0,+\infty), \tag{1.1}$$

is considered. Here $L : D^k_{loc}[0,\infty) \to L_{loc}[0,\infty)$ and $T : L_{loc}[0,\infty) \to L_{loc}[0,\infty)$ are linear Volterra operators. $D^k_{loc}[0,\infty)$ is a space of functions $z : [0,\infty) \to R^1$ absolutely continuous with derivative of $(k-1)-$th order on each finite interval, and $L_{loc}[0,\infty)$ is a space of locally summable functions $w : [0,\infty) \to R^1$. It should be noted that the operators L and T act on $v(\cdot,x)$ or on $v''_{xx}(\cdot,x)$ respectively as on the functions of a variable t only for fixed x. It is also assumed that the operators L and T cannot depend on x and do not include derivatives with respect to x.

Obviously, the classical wave, heat and Laplace equations can be written in the form of equation (1.1), which also includes natural "delay" and "integro-differential" generalizations of the classical equations. For example, the heat equation with delay

$$v'_t(t,x)(t) = v''_{xx}(t-\theta(t),x), \tag{1.2}$$

and the wave equation with delay

$$v''_{tt}(t,x)(t) = v''_{xx}(t-\theta(t),x), \tag{1.3}$$

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can be considered. The theory and various applications of partial functional differential equations are presented in the monograph by J. H. Wu [14]. "Integro-differential" generalizations of the classical equations is another possible realization of equation (1.1). Refer in this connection to the well-known monograph by J. Pruss [12] in which many mathematical models containing integral partial differential equations (PDE) are studied.

Let us write several possible forms of the operators L and T:

$$Lv(\cdot,x)(t) \equiv v_t^{(k)}(t,x) + \sum_{j=0}^{r}\sum_{i=1}^{m} p_{ji}(t) v_t^{(j)}(t-\tau_{ji}(t),x)$$

$$+ \sum_{j=0}^{r} \int_0^t K_j(t,\xi) v_t^{(j)}(\xi,x)d\xi, \;\; r<k,$$

$$Tv''_{xx}(\cdot,x)(t) \equiv \sum_{i=1}^{m} a_i(t) v''_{xx}(t-\theta_i(t),x) + \int_0^t Q(t,\xi) v''_{xx}(\xi,x)d\xi,$$

where

$$v_t^{(i-1)}(\xi,x) = \varphi_i(\xi,x),\; i=1,\ldots,k,\;\; v''_{xx}(\xi,x) = \varphi_0(\xi,x), \quad x\in[0,\omega], \quad \xi<0. \tag{1.4}$$

where p_{ji}, a_i, K_j, Q are continuous functions, $\tau_{ji}(t)$ and $\theta_i(t)$ are measurable positive functions, $\varphi_i(\xi,x), \varphi_0(\xi,x)$ are continuous functions $(i=1,\ldots,m,\; j=1,\ldots,r)$.

This paper deals with **oscillation** and **asymptotic properties** of solutions to the Dirichlet boundary value problem (1.1), (1.5), where

$$v(t,0) = v(t,\omega) = 0, \;\; t\in[0,+\infty). \tag{1.5}$$

Definition 1.1. We say that a solution $u(t,x)$ of the PDE boundary value problem oscillates if for each t_0 there exists a point $(t_1,x_1): t_1>t_0,\; x_1\in(0,\omega)$ such that $u(t_1,x_1)=0$.

Below we assume that there exist solutions to the boundary value problem in order not to discuss the question of solution existence here.

In this paper the PDE boundary value problem is reduced to an "ordinary delay equation". The properties of this equation allow us to make a conclusion about the oscillation and asymptotic behavior of solutions to the considered PDE problem.

Oscillation properties of hyperbolic equations were considered by P. Wang in the paper [13].

The following partial differential-difference equation

$$v(p(t),x) + a(t)v''_{xx}(t,x) + b(t,x)v(r(t),x) = 0, \;\; t>0, \;\; x\in[0,\omega], \tag{1.6}$$

where p and r are monotone increasing functions such that $p(t)\ge t,\; r(t)\le t$, was considered in the paper by D. Bainov and Yu. Domshlak [3]. In that case the operator L is a functional operator of the form

$$Lv(\cdot,x)(t) \equiv v(p(t),x) + b(t,x)v(r(t),x), \quad t>0, \quad x\in[0,\omega]. \tag{1.7}$$

Also, in [3], estimates of the zone of solutions positivity for the Dirichlet boundary value problem with equation (1.6) were derived. Because of the instability of equation (1.1) (see, Theorem 2.4) it is impossible to use the passage to the limit and to tend directly the results on oscillation and asymptotic properties from differential-difference equation (1.6) to the PDE (1.1).

Thus estimates of the domains of solutions positivity will be obtained in terms of the spectral radius of the compact operator

$$(K_{\nu,\mu}z)(t) = -\int_{\nu}^{\mu} G_{\nu,\mu}(t,s)\sum_{i=0}^{m} p_i(s)z(s-\tau_i(s))ds, \quad t\in[\nu,\mu], \tag{1.8}$$

acting in the space of continuous functions. Here $z(s)=0$ if $s<\nu$, $p_0(t) = a(t)(\frac{n\pi}{\omega})^2$, $\tau_0(t)=\theta(t)$, and $G_{\nu,\mu}(t,s)$ is the Green function of the two point boundary value problem

$$z''(t) = f(t), \quad t\in[\nu,\mu], \;\; z(\nu)=0, \; z(\mu)=0. \tag{1.9}$$

Denote by $r_{\nu,\mu}$ the spectral radius of the operator $K_{\nu,\mu}$.

Various estimates of the distance between adjoint zeros of oscillating solutions for “ordinary delay equations” were obtained by N. V. Azbelev [1], Yu. Domshlak [8], A. D. Myshkis [10], S. B. Norkin [11] and in the work [5]. In Part 2 we propose an assertion about an estimate of the zone of solution positivity for the PDE (1.1).

Results on the unboundedness of solutions of “ordinary delay equations” were obtained by the author in a recent paper [6] with a comprehensive bibliography. In Part 2 an assertion about the unboundedness of solutions of the PDE (1.3) is given.

In order to illustrate our results on this subject, let us consider the equation

$$v''_{tt}(t,x) - v''_{xx}(t-\varepsilon,x) = 0, \; t\in[0,+\infty), \quad x\in[0,\omega]. \tag{1.10}$$

If $\varepsilon>0$, then there exist unbounded solutions of the Dirichlet problem (1.10), (1.5). If $\varepsilon=0$, then all solutions of this problem are bounded on $(0,+\infty)$.

2. Main Results

Let us consider the Dirichlet boundary value problem for the heat equation

$$\begin{gathered} v'_t(t,x) = v''_{xx}(t-\theta,x), \;\; t>0, \;\; x\in[0,\omega], \\ v(t,0)=v(t,\omega)=0, \quad v(0,x)=\alpha(x), \end{gathered} \tag{2.1}$$

where θ is a positive constant, $\alpha(x)$ is a continuous function such that $\alpha(0) = \alpha(\omega) = 0$ and $v''_{xx}(s,x)=0$ for $s<0$.

For problem (2.1) in the case of the classical heat equation ($\theta=0$) the maximum principle is valid, solutions are positive when $\alpha(x)>0$ and tend to zero when $t\to\infty$. If the delay θ is not zero, then, in contrast to these facts, we have the following assertion.

Theorem 2.1. *There exist unbounded solutions of the Dirichlet boundary value problem* (2.1) *and the maximum principle is not valid for the heat equation. If*

$$\theta > \frac{1}{e}\,\frac{\omega^2}{\pi^2}, \tag{2.2}$$

then all solutions of this problem oscillate.

Remark 2.1. Inequality (2.2) guarantees that in a narrow enough zone all solutions oscillate.

Let us consider the following particular case of equation (1.1):

$$Lv(\cdot,x)(t) - a(t)v''_{xx}(t-\theta(t),x) = 0, \quad x \in [0,\omega], \quad t \geqslant 0, \tag{2.3}$$

where

$$v_t^{(i-1)}(\xi,x) = 0, \;\; i = 1,\dots,k, \quad v''_{xx}(\xi,x) = 0, \;\; x \in [0,\omega], \;\; \xi < 0. \tag{2.4}$$

Theorem 2.2. 1) *If for any n there exist unbounded solutions of the "ordinary delay equation"*

$$Lz(t) + \left(\frac{n\pi}{\omega}\right)^2 a(t)z(t-\theta(t)) = 0, \quad t \geqslant 0, \tag{2.5}$$

then there exist unbounded solutions of the Dirichlet problem (2.3), (1.5).

2) *If each solution of the equation*

$$Lz(t) + \left(\frac{\pi}{\omega}\right)^2 a(t)z(t-\theta(t)) = 0, \quad t \geqslant 0, \tag{2.6}$$

oscillates, then each solution of the Dirichlet problem (2.3), (1.5) *oscillates.*

Let us consider the following particular case of equation (2.3):

$$v''_{tt}(t,x) - a(t)v''_{xx}(t-\theta(t),x) = 0, \quad t > 0, \quad x \in [0,\omega]. \tag{2.7}$$

Corollary 2.1. *If θ is a bounded function on $[0,+\infty)$ and there exists a positive constant ε such that*

$$a(t) \geq \frac{(1+\varepsilon)}{4t^2}\,\frac{\omega^2}{\pi^2}, \;\; for \;\; t > b > 0, \tag{2.8}$$

then all solutions of the Dirichlet problem (2.7), (1.5) *oscillate on* $(0,+\infty)\times(0,\omega)$.

Remark 2.2. Inequality (2.8) cannot be improved in the following sense. If we consider the classical hyperbolic equation ($\theta = 0$) and set $\varepsilon = 0$ in inequality (2.8), then there exists a positive solution $v(t,x) = \sqrt{t}\sin(\frac{\pi}{\omega}x)$ of the Dirichlet problem

$$v''_{tt}(t,x) - \frac{1}{4t^2}\frac{\omega^2}{\pi^2}v''_{xx}(t,x) = 0, \quad t > b, \quad x \in [0,\omega],$$
$$v(t,0) = v(t,\omega) = 0, \quad t \in [b,+\infty).$$

Remark 2.3. If the coefficient $a(t)$ in equation (2.7) is of the form $a(t) = \frac{C}{t^2}$, then the geometrical size ω of a zone influences the oscillation of solutions. Inequality (2.8) guarantees that in a narrow enough zone all solutions oscillate.

We propose an approach reducing study of oscillation to the classical problem about the spectral radius of the compact operator $K_{\nu,\mu}$ determined by formula (1.8).

Let us determine $\nu^* = \operatorname{vrai\,inf}_{t\in[\nu,\mu]} h(t)$, where $h(t) = \min_{1\le i\le m}\{t-\tau_i(t)\}$.

Theorem 2.3. *Let the inequality $r_{\nu,\mu} \ge 1$ be fulfilled. Then all nontrivial solutions of the Dirichlet problem for the equation*

$$v''_{tt}(t,x) - a(t)v''_{xx}(t,x) + \sum_{i=1}^{m} p_i(t)v(t-\tau_i(t),x) = 0, \quad t>0, \quad x\in[0,\omega], \quad (2.9)$$

change their sign in the zone $(\nu^*,\mu)\times(0,\omega)$.

Denote $\tau^* = \sup_{t\in[\beta,\mu]} \max_{1\le i\le m} \tau_i(t)$ and $a_0 = \inf_{t\in[\beta,\mu]} a(t)$

Corollary 2.2. *All nontrivial solutions of problem* (2.9), (1.5) *change their sign in a zone $(\beta,\mu)\times(0,\omega)$ if*

$$\mu - \beta > \frac{\omega}{\sqrt{a_0}} + \tau^*. \quad (2.10)$$

Remark 2.4. Inequality (2.10) cannot be improved in the following sense. If instead of the inequality we put the equality in (2.10), then the assertion of Corollary 2.2 is not true. Let us consider the following example: a is a positive constant and $p_i = 0$ for $i = 1, 2, \ldots, m$. It is clear that $\tau^* = 0$ and the function

$$v(t,x) = \sin[\sqrt{a}\frac{\pi}{\omega}(t-\beta)]\sin\frac{\pi}{\omega}x \quad (2.11)$$

is a solution which is positive in the zone $(\beta, \beta+\frac{\omega}{\sqrt{a}})\times(0,\omega)$.

The following assertion shows whether there exist unbounded solutions of PDE boundary value problems.

Theorem 2.4. *If $a(t)$ and $t-\theta(t)$ are nondecreasing functions, $a(t)$ is a bounded function and $\int^{\infty}\theta(t)dt = \infty$, then there exist unbounded solutions of the Dirichlet problem* (2.7), (1.5).

3. Proofs

Proof of Theorem 2.2. Let us denote

$$z(t) = \int_0^{\omega} \sin\left(\frac{\pi n}{\omega}x\right) v(t,x)dx, \quad t\in[0,+\infty), \quad (3.1)$$

Multiplying each term in equation (2.3) by $\sin(\frac{\pi n}{\omega}x)$ and integrating, we get the following equation for the function z :

$$(Lz)(t) + \left(\frac{n\pi}{\omega}\right)^2 a(t)z(t-\theta(t)) = 0, \quad t\in[0,+\infty), \quad (3.2)$$

where $z(s) = 0$ for $s < 0$.

If $n = 1$, then

$$z(t) = \int\limits_0^\omega \sin(\frac{\pi}{\omega}x)v(t,x)dx, \quad t \in [0,+\infty), \tag{3.3}$$

is positive for a positive function $v(t,x)$, and if $z(t)$ oscillates, then $v(t,x)$ also oscillates. It proves assertion 2).

From formula (3.1) it follows that for each bounded $v(t,x)$ the function $z(t)$ is also bounded. If for any n the corresponding function $z(t)$ is unbounded, this implies that $v(t,x)$ is unbounded too. It proves assertion 1). □

Proof of Corollary 2.1. Multiplying each term in equation (2.7) by $\sin(\frac{\pi}{\omega}x)$ and integrating, we get the following equation for the function z :

$$z''_{tt}(t,x) + a(t)\frac{\pi^2}{\omega^2}z(t-\theta(t)) = 0, \quad t > 0, \tag{3.4}$$

The inequality

$$p(t) \geq \frac{1+\varepsilon}{4t^2}, \quad \varepsilon > 0,$$

is one of the classical tests of oscillation of the ordinary differential equation

$$z''_{tt}(t) + p(t)z(t) = 0, \quad t > 0.$$

Now it is clear that the inequality

$$a(t) \geq \frac{(1+\varepsilon)}{4t^2}\,\frac{\omega^2}{\pi^2} \quad \text{for} \quad t > 0, \quad \varepsilon > 0, \tag{3.5}$$

guarantees the oscillation of the ordinary differential equation

$$z''_{tt}(t) + a(t)\frac{\pi^2}{\omega^2}z(t) = 0, \quad t > 0. \tag{3.6}$$

The known theorem of Brands [4] claims that the one term second order equation (3.4) with bounded delay θ is oscillatory iff the corresponding ordinary equation (3.6) is oscillatory. Now oscillation of solutions to equation (3.4) follows from condition (3.5). Now assertion 2) of Theorem 2.2 completes the proof. □

Proof of Theorem 2.4. Multiplying each term in equation (2.7) by $\sin(\frac{\pi}{\omega}x)$ and integrating, we get equation (3.4) for the function z. By Theorem 1.1 from [6] there exist unbounded solutions to equation (3.4). Now assertion 1) of Theorem 2.2 completes the proof. □

Proof of Theorem 2.1. Let us introduce $z(t)$ by formula (3.1). Multiplying each term in equation (2.1) by $\sin(\frac{\pi n}{\omega}x)$ and integrating, we get the following equation for $z(t)$:

$$z'(t) + \left(\frac{\pi n}{\omega}\right)^2 z(t-\theta) = 0, \quad t \in [0,+\infty). \tag{3.7}$$

It is clear that $\left(\frac{n\pi}{\omega}\right)^2 \to +\infty$ as $n \to \infty$ and for positive θ we obtain the existence of unbounded solutions of this equation. Condition (2.2) implies oscillation of solutions of equation (3.7) for $n = 1$. The assertions of Theorem 2.1. follow now from Theorem 2.2. □

Proof of Theorem 2.3 is based on the following assertion proved in [7].

Lemma 3.1 ([7]). *If* $r_{\nu,\mu} \geq 1$, *then each nontrivial solution of the "ordinary delay equation"*

$$z'(t) + a(t)\left(\frac{\pi}{\omega}\right)^2 z(t-\theta) + \sum_{i=1}^{m} p_i(t) z(t-\tau_i(t)) = 0, \quad t \in [0, +\infty), \tag{3.8}$$

changes its sign in the interval (ν^*, μ).

Multiplying each term in equation (2.9) by $\sin(\frac{\pi}{\omega}x)$ and integrating, we get equation (3.8) for the function z. The proof of Theorem 2.3 follows now from formula (3.3). □

Proof of Corollary 2.2 is based on the estimate of the spectral radius $r_{\nu,\mu} \geq 1$ of the compact operator $K_{\nu,\mu}$ obtained in [7]. Multiplying each term in equation (2.9) by $\sin(\frac{\pi}{\omega}x)$ and integrating, we get equation (3.8) for the function z. Condition (2.10) implies the estimate $r_{\nu,\mu} \geq 1$ of the spectral radius. □

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