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# EQUILIBRIUM FOR PERTURBATIONS OF MULTIFUNCTIONS BY CONVEX PROCESSES

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Abstract. We present a general equilibrium theorem for the sum of an upper hemicontinuous convex valued multifunction and a closed convex process defined on a noncompact subset of a normed space. The lack of compactness is compensated by inwardness conditions related to the existence of viable solutions of some differential inclusion.

## 1. Introduction and Preliminaries

Equilibrium theorems provide sufficient conditions for the existence of an *equilibrium* (or a zero) for a given multifunction $\Phi$ under certain constraints, that is, a solution to the inclusion $0 \in \Phi(x)$ required to belong to a certain constraint set $X$.

Many important problems in nonlinear analysis can be reduced to equilibrium problems (for example, the problem of existence of critical points for smooth and non-smooth functions, the problem of existence of stationary solutions to differential inclusions, etc.). In the mid-seventies B. Cornet [1] derived an equilibrium theorem for multifunctions defined on compact convex constraint sets from classical results of Ky Fan and F. Browder by using the celebrated inf-sup inequality of Ky Fan [2] (see, for instance, Aubin [3] and Aubin and Frankowska [4] and references therein).

The purpose of this paper is to present an equilibrium theorem for the sum of an upper hemicontinuous multifunction with closed convex values and a closed convex process defined on a closed convex subset of a normed space (Theorem 3.1 below). The lack of compactness of the domain $X$ is compensated by tangency conditions of the Ky Fan-type on a compact subset $K$ of $X$ and outside of it. These tangency conditions are necessary for the

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existence of viable solutions to a general differential inclusion. Our equilibrium theorem generalizes a classical result of G. Haddad [5] on the existence of a stationary point of differential inclusions to noncompact viability domains. It contains as particular cases results in [1], [4], Ben-El-Mechaiekh [6], Bae and Park [7], Simons [8], and others.

All topological spaces in this paper are assumed to be Hausdorff spaces. The *closure* and the *boundary* of a subspace $A$ of a topological space are denoted by $\overline{A}$ and $\partial A$ respectively. The *open ball* of radius $\epsilon > 0$ around a subset $A$ of a normed space $E$ is denoted by $B_E(A, \epsilon)$.

Locally convex real topological vector spaces are simply called *locally convex spaces*. The *topological dual* of a topological vector space $E$ is denoted by $E^*$ and the *convex hull* of a subset $K$ of $E$ is denoted by $\mathrm{co}(K)$.

Given a subset $X$ of a topological vector space $E$ and an element $x \in E$, $S_X(x)$ denotes the cone $\bigcup_{t>0} \frac{1}{t}(X-x)$ spanned by $X-x$, and $N_X(x)$ denotes the set $\{\varphi \in E^* : \langle \varphi, x\rangle \geq \sup_{v\in X}\langle \varphi, v\rangle\}$. Observe that if $E$ is a normed space, $X$ is convex in $E$, and $x \in X$, then $\overline{S_X(x)}$ is precisely the tangent cone to $X$ at $x$ and $N_X(x)$ is the negative polar cone of $\overline{S_X(x)}$, called the *normal cone* to $X$ at $x$.

Given a subset $K$ of a topological vector space $E$, the *support function* of $K$ is the function $\sigma(K, .) : E^* \longrightarrow R \cup \{+\infty\}$ defined as

$$\sigma(K, \varphi) := \sup_{y\in K} \langle \varphi, x\rangle, \quad \varphi \in E^*. \tag{1}$$

We recall that the *negative* (respectively *positive*) *polar cone* of $K$ is the set

$$K^- := \{\varphi \in E^* : \sigma(K, \varphi) \leq 0\} \quad (K^+ = -K^- \text{ respectively}). \tag{2}$$

Observe that given two subsets $K_1, K_2$ of $E$ and $\varphi \in E^*$,

$$\sigma(K_1 + K_2, \varphi) = \sigma(K_1, \varphi) + \sigma(K_2, \varphi). \tag{3}$$

In particular, if $K_2$ is a cone, then $\sigma(K_1 + K_2, \varphi) = \sigma(K_1, \varphi)$ if $\varphi \in K_2^-$ and $+\infty$ otherwise.

By the Hahn–Banach separation theorem, the closed convex hull of $K$ is characterized by

$$\overline{\mathrm{co}(K)} = \{x \in E : \langle \varphi, x\rangle \leq \sigma(K, \varphi), \forall \varphi \in E^*\}. \tag{4}$$

A *multifunction* $\Phi$ from a set $X$ into a set $F$ is a map from $X$ into the family $\mathcal{P}(F)$ of all subsets of $F$. The multifunction $\Phi$ is said to be *strict* if all of its values are nonempty. The *domain* of $\Phi$ is the set of all those elements $x \in X$ for which $\Phi(x)$ is nonempty. If $F$ is a vector space, an element $x_0 \in X$ is said to be an *equilibrium* for a multifunction $\Phi : X \longrightarrow \mathcal{P}(F)$ if $0 \in \Phi(x_0)$.

This paper is particularily concerned with two types of multifunctions, namely *upper hemicontinuous* multifunctions with convex values and *closed convex processes*. Upper hemicontinuity is a weak form of upper semicontinuity, and closed convex processes (introduced by Rockafellar [9]) are multivalued analogues of linear operators. For details concerning these two concepts, we refer to [4].

**Definition 1.1.** A multifunction $\Phi : X \longrightarrow \mathcal{P}(F)$ from a topological space $X$ into a topological vector space $F$ is said to be upper hemicontinuous at $x_0 \in X$ if for any $\psi \in F^*$, the function $x \longmapsto \sigma(\Phi(x), \psi)$ is upper semicontinuous at $x_0$. It is said to be upper hemicontinuous on $X$, if and only if it is upper hemicontinuous at each point of $X$.

Any upper semicontinuous multifunction from $X$ into $F$ supplied with the weak topology is upper hemicontinuous. Conversely, if $\Phi$ is upper hemicontinuous at a point $x_0$ and if $\Phi(x_0)$ is convex and weakly compact, then $\Phi$ is also upper semicontinuous at $x_0$. We recall that a finite sum and a finite product of upper hemicontinuous multifunctions is upper hemicontinuous.

**Definition 1.2.** (i) A closed convex process $\Lambda$ from a locally convex space $E$ into a locally convex space $F$ is a multifunction $\Lambda : E \longrightarrow \mathcal{P}(F)$ whose graph is a closed convex cone.

(ii) The transpose of a closed convex process $\Lambda : E \longrightarrow \mathcal{P}(F)$ is the closed convex process $\Lambda^* : F^* \longrightarrow \mathcal{P}(E^*)$ defined by

$$\varphi \in \Lambda^*(\psi) \iff \langle \varphi, x\rangle \leq \langle \psi, y\rangle, \ \forall x \in E, \forall y \in \Lambda(x). \tag{5}$$

Important examples of closed convex processes are provided by contingent derivatives of multifunctions.

If the domain of $\Lambda$ is the whole space $E$, then the domain of $\Lambda^*$ is the positive polar cone $\Lambda(0)^+$.

A *linear process* is a multifunction whose graph is a vector subspace.

The support functions of a closed convex process $\Lambda : E \longrightarrow \mathcal{P}(F)$ and of its transpose $\Lambda^*$ are related by the following

**Lemma 1.3 (see [4]).** *For every $\psi_0$ in the interior of the domain of $\Lambda^*$ and for every $x_0$ in the barrier cone $b(\Lambda^*(\psi_0)) := \{x \in E : \sigma(\Lambda^*(\psi_0), x) < +\infty\}$ of $\Lambda^*(\psi_0)$, there exists $y_0 \in \Lambda(x_0)$ such that*

$$\langle \psi_0, y_0\rangle = \sigma(\Lambda^*(\psi_0), x_0) = -\sigma(\Lambda(x_0), -\psi_0). \tag{6}$$

Let $E$ and $F$ be two normed spaces. Denote by $\Lambda(E, F)$ the normed space of all closed convex processes from $E$ into $F$ equipped with the norm

$$\|\Lambda\| := \sup_{u \in E\setminus\{0\}} \inf_{v \in \Lambda(u)} \frac{\|v\|}{\|u\|}.$$

The uniform boundedness principle holds true for convex processes between normed spaces (again, see [4]); it implies the following crossed convergence property.

Let us recall first that a mapping $\Lambda : X \longrightarrow \Lambda(E,F)$ is said to be *pointwise bounded* if and only if

$$\forall u \in E, \exists v_x \in \Lambda(x)(u) \text{ with } \sup_{x\in X} \|v_x\| < +\infty. \tag{7}$$

**Lemma 1.4 ([10]).** *Let $X$ be a topological space, $E, F$ be normed spaces, and $\Lambda : X \longrightarrow \Lambda(E,F)$ be pointwise bounded. Then the following conditions are equivalent:*

(a) *the multifunction $x \longmapsto \operatorname{graph}(\Lambda(x))$ is lower semicontinuous;*

(b) *the multifunction $(x,u) \longmapsto \Lambda(x)(u)$ is lower semicontinuous.*

## 2. Tangency in the sense of Ky Fan

Let $E, F$ be two topological vector spaces, $X$ be a subset of $E$, $\Phi : X \longrightarrow \mathcal{P}(F)$ be a multifunction with domain $X$, and $\Lambda : X \longrightarrow \Lambda(E,F)$ be a mapping.

**Definition 2.1.** Given a subset $X_1$ of $X$ and a subset $X_2$ of $E$, we say that the multifunction $\Phi$ satisfies the condition of Ky Fan on $(X_1, X_2)$ with respect to $\Lambda$ if and only if

$$\overline{\Lambda(x)(-S_{X_2}(x))} \cap \Phi(x) \neq \varnothing, \quad \forall x \in X_1. \tag{8}$$

In other terms, the directions of $\Phi$ on $X_1$ are controlled by a family of cones depending on $X_1$ and $X_2$. Weaker forms of this tangency condition were considered by various authors (e.g., [3], [6, [11], [8]). In the case where $E = F$, $\Lambda(x) \equiv -Id_E, X_1 = X_2 = X$, (8) reads as

$$\Phi(x) \cap \overline{S_X(x)} \neq \varnothing, \quad \forall x \in X,$$

that is, the multifunction $\Phi$ is *inward* in the sense of Ky Fan [12].

**Lemma 2.2.** *If the multifunction $\Phi$ satisfies the condition of Ky Fan on $(X_1, X_2)$ with respect to $\Lambda$, then*

$$\inf\{\sigma(\Phi(x),\psi) : \psi \in F^*, \Lambda(x)^*(\psi) \cap N_{X_2}(x) \neq \varnothing\} \geq 0. \tag{9}$$

*The converse holds true whenever the values of $\Phi$ are compact.*

*Proof.* Let $x \in X_1$ be arbitrary and let $\psi \in F^*$ be such that $\Lambda(x)^*(\psi) \cap N_{X_2}(x) \neq \varnothing$. Choose an element $y \in \Phi(x) \cap \overline{\Lambda(x)(-S(x))}$, that is, $y$ is the limit of a net $\{y_i\}$ with $y_i \in \Lambda(x)(v_i), -v_i = \frac{1}{t_i}(x_i - x) \in S_{X_2}(x), x_i \in X_2, t_i > 0$.

Clearly, $\forall \varphi \in \Lambda(x)^*(\psi) \cap N_{X_2}(x)$, $\langle \varphi, x\rangle \geq \langle \varphi, x_i\rangle$ and thus $\langle \varphi, v_i\rangle \geq 0$, $\forall i$. By (5) $\langle \psi, y_i\rangle \geq \langle \varphi, v_i\rangle \geq 0$, $\forall i$. Hence $\sigma(\Phi(x),\psi) \geq \langle \psi, y\rangle \geq 0$.

Conversely, given $x \in X_1$, if $\Phi(x)$ is compact, then the set $\Phi(x) - \overline{\Lambda(x)(-S_{X_2}(x))}$ is closed and convex. Assume that (9) is satisfied and let $\psi \in F^*$ be such that $\Lambda(x)^*(\psi) \cap N_{X_2}(x) \neq \varnothing$ and $\sigma(\Phi(x), \psi) \geq 0$. Observe that $\Lambda(x)^*(\psi) \cap N_{X_2}(x) \neq \varnothing$ is equivalent to $\psi \in \Lambda(x)^{*-1}(N_{X_2}(x)) = \Lambda(x)^{*-1}(S_{X_2}(x)^-) = \Lambda(x)^{*-1}(-S_{X_2}(x)^+)$ which, by the bipolar theorem (Theorem 2.5.7 in [4]), equals $-[\Lambda(x)(-S_{X_2}(x))]^- = -\overline{\Lambda(x)(-S_{X_2}(x))}^-$. By the remark following (3), $\sigma(\Phi(x) - \overline{\Lambda(x)(-S_{X_2}(x))}, \psi) = \sigma(\Phi(x), \psi)$. The characterization (4) ends the proof. $\square$

Condition (8) is necessary for the solvability of the following differential inclusion:

$$\Lambda(x'(t)) \subset \Phi(x(t)), \quad t \in [0, T], \tag{10}$$

where $\Lambda : R^n \longrightarrow \mathcal{P}(R^m)$ is a linear process, and $\Phi : R^n \longrightarrow \mathcal{P}(R^m)$ is an upper hemicontinuous multifunction with closed convex values. Let $K$ be a subset of $R^n$ containing the initial point $x(0) = x_0$.

**Proposition 2.3.** *If $x(.)$ is a solution of problem* (10) *satisfying the following condition:*

$$\forall T' \in (0, T], \exists t \in (0, T'] \text{ such that } x(t) \in K, \tag{11}$$

*then $\overline{\Lambda(S_K(x_{0)})} \cap \Phi(x_0) \neq \varnothing$, that is, $-\Phi$ satisfies the condition of Ky Fan on $(\{x_0\}, K)$ with respect to $\Lambda$.*

*Proof.* By (11) there exists a sequence of positive reals $\{t_k\}_{k \in N}$ converging to $0^+$ such that $x(t_k) \in K$. Since $\forall \psi \in R^m$, the real function $x \longmapsto \sigma(\Phi(x), \psi)$ is upper semicontinuous, then $\forall \epsilon > 0, \exists \delta_\psi > 0$ such that

$$\sigma(\Lambda(x'(\tau)), \psi) \leq \sigma(\Phi(x(\tau)), \psi) < \sigma(\Phi(x_0), \psi) + \epsilon\|\psi\|, \quad \forall \tau \in [0, \delta_\psi].$$

The definition of the transpose of a closed convex process (5) implies that

$$\forall \tau \in [0, \delta_\psi], \langle \varphi, x'(\tau) \rangle < \sigma(\Phi(x_0), \psi) + \epsilon\|\psi\|, \quad \forall \varphi \in \Lambda^*(\psi).$$

Hence,

$$\begin{aligned} \forall k, \forall \varphi \in \Lambda^*(\psi), \frac{1}{t_k} \int_0^{t_k} \langle \varphi, x'(\tau) \rangle d\tau &\leq \frac{1}{t_k} \int_0^{t_k} [\sigma(\Phi(x_0), \psi) + \epsilon\|\psi\|] d\tau \\ &= \sigma(\Phi(x_0), \psi) + \epsilon\|\psi\|. \end{aligned}$$

We conclude that $\sigma(\Lambda^*(\psi), v_k\rangle \leq \sigma(\Phi(x_0), \psi) + \epsilon\|\psi\|, v_k = \frac{1}{t_k}(x(t_k) - x_0)$, $\forall k$. By Lemma 1.3 and since the domain of $\Lambda^* = \Lambda(0)^+ = \{0\}^+ = R^m, \forall k$, $\exists y_k \in \Lambda(v_k)$ such that $\langle \psi, y_k \rangle = \sigma(\Lambda^*(\psi), v_k\rangle$. Being bounded by the uniform boundedness principle, the sequence $\{y_k\}_{k \in N}$ converges to some $y \in \overline{\Lambda(S_K(x_0))}$ satisfying the inequality $\langle \psi, y \rangle \leq \sigma(\Phi(x_0), \psi) + \epsilon\|\psi\|$. Since

$\epsilon$ and $\psi$ are arbitrary, (4) implies that $y \in \Phi(x_0)$. Finally, observe that since $\Lambda(0) = \{0\}$, then $\overline{\Lambda(S_K(x_0))} \cap \Phi(x_0) \neq \varnothing$ implies that $\overline{\Lambda(-S_K(x_0))} \cap -\Phi(x_0) \neq \varnothing$. $\square$

*Remark.* If for all $x_0 \in K$, there is a viable trajectory $x(.)$ of the differential inclusion (10) in $K$ (that is, $x(t) \in K$, $\forall t \in [0,T]$) starting at $x_0$, then (11) is obviously satisfied and therefore $-\Phi$ satisfies the condition of Ky Fan on $(K,K)$ with respect to $\Lambda$.

We will see later (Corollary 3.2 below) that one of the consequences of our main theorem is the existence of a stationary solution of (10) for noncompact viability domains.

## 3. The Main Theorem

We state now our main theorem.

**Theorem 3.1.** *Let $X$ be a convex subset of a normed space $E, F$ be a normed space, $\Phi : X \longrightarrow \mathcal{P}(F)$ be a strict upper hemicontinuous multifunction with closed convex values, and $\Lambda : X \longrightarrow \Lambda(E,F)$ be a continuous mapping satisfying the boundedness condition:*

(i) $\exists M > 0$ *such that* $\forall x \in X$, $\forall u \in E$ *with* $\|u\| = 1$, $\exists v \in \Lambda(x)(u)$ *such that* $\|v\| \leq M$.

*Furthermore, assume that there exists a compact subset $K$ of $X$ such that:*

(ii) *for each finite subset $N$ of $X$, there exists a compact convex subset $C_N$ of $X$ containing $N$ such that $\Phi$ satisfies the condition of Ky Fan on $(C_N \setminus K, C_N)$ with respect to $\Lambda$;*

(iii) $\Phi$ *satisfies the condition of Ky Fan on $(K,X)$ with respect to $\Lambda$.*

*Then,*

(A) $\Phi$ *has an equilibrium;*

(B) $\forall x_0 \in X$, *the multifunction $\Phi(.) + \Lambda(.)(. - x_0)$ has an equilibrium.*

*Remarks.* (**1**) This theorem remains valid in the context of spaces having separating duals (e.g., locally convex spaces) or convex spaces in the sense of [13].

(**2**) If $\forall x \in X$, $\Lambda(x)$ is a linear process, then (B) is the coincidence property:

$$\forall x_0 \in X, \exists \hat{x} \in X \text{ such that } \Lambda(\hat{x})(x_0) \cap \Phi(\hat{x}) + \Lambda(\hat{x})(\hat{x}) \neq \varnothing.$$

(**3**) In the case where $X$ is compact, (i) follows from the continuity of the operator $\Lambda$, and putting $K = \varnothing$ and $C_N = X$ for any $N$, (ii) and (iii) reduce to:

(ii)$'$ $\Phi$ satisfies the condition of Ky Fan on $(X,X)$ with respect to $\Lambda$; that is the direction of the multifunction $\Phi$ is controlled by a family of convex cones. In this case, if for each interior point $x \in X$, $\Lambda(x)$ is a surjective process, (ii)$'$ is simply the tangency condition:

$$\Phi \text{ satisfies the condition of Ky Fan on } (\partial X, X) \text{ with respect to } \Lambda.$$

When $X$ is compact and $\Lambda(x) = \ell(x), x \in X$, is a bounded linear operator from $E$ into $F$ this result is precisely the solvability theorem in [4].

(**4**) In the case where $\forall x \in X$, $\Lambda(x) \equiv \ell$ is a bounded linear operator from $E$ into $F$, and $C_N = C$ is the same for all finite subsets $N$ of $X$, this result can be found in [6]. In this case (iii) is equivalent to:

$$\Phi \text{ satisfies the condition of Ky Fan on } (K \cap \partial X, X) \text{ with respect to } \ell,$$

and (B) guarantees the surjectivity of the perturbation $\Phi + \ell$ onto $\ell(X)$.

(**5**) In order to formulate our next result, we will consider a weaker form of condition (ii) involving the concept of $c$-compactness of [13] in our main theorem (for a particular case of this particular instance, see [7]). Recall that a subset $C$ of $X$ is said to be $c$-compact in $X$ if for each finite subset $N$ of $X$ there exists a compact convex subset $C_N$ of $X$ such that $C \cup N \subset C_N$. Note that any bounded subset of a finite dimensional space is $c$-compact. Condition (ii) in Theorem 8 may be replaced by

$$\text{(ii)}'' \ \exists C \ c\text{-compact } \subseteq X \text{ such that } \overline{\Lambda(-S_{\mathrm{co}(\{x\} \cup C)}(x)} \cap \Phi(x) \neq \varnothing, \quad \forall x \in X \setminus K.$$

As an immediate consequence of Remark 5 above, we obtain that existence of viable solutions in a noncompact domain implies the existence of a stationary solution (or rest point) of the inclusion (10). More precisely, we have

**Corollary 3.2.** *Let $X$ be a closed convex subset of $R^n, K$ a compact subset of $X, C$ a c-compact subset of $X$, $\Lambda : R^n \longrightarrow \mathcal{P}(R^m)$ a strict linear process, and $\Phi : X \longrightarrow \mathcal{P}(R^m)$ a strict upper hemicontinuous multifunction with closed convex values.*

*Assume that the following properties are satisfied:*

(i) $\forall x_0 \in K$, $\exists x(.)$ *solution of* (10) *starting at* $x_0$, *such that* $\forall T' \in (0, T]$, $\exists t \in (0, T']$ *with* $x(t) \in X$;

(ii) $\forall x_0 \in X \setminus K, \exists x(.)$ *solution of* (10) *starting at* $x_0$, *such that* $\forall T' \in (0, T]$, $\exists t \in (0, T']$ *with* $x(t) \in \mathrm{co}(\{x_0\} \cup C)$;

*Then $\Phi$ has an equilibrium in $X$.*

*Remarks.* (**1**) Conditions (i)–(ii) state that if a trajectory of (10) starts in $K$ then it must first enter $X$, and if it starts in $X \setminus K$ then it is first attracted by $C$ in the sense that the trajectory must first enter the drop with vertex at the initial point and base $C$.

(**2**) When $X$ is a compact convex viability domain of $\Phi$, $R^n = R^m$ and $\Lambda = Id_{R^n}$, then (i) and (ii) are obviously satisfied with $K = C = X$ and we retrieve the equilibrium theorem of [4].

## 4. Proof of the Main Theorem

The starting point is the following generalization of the Browder–Ky Fan fixed point theorem for convex-valued multifunctions with open fibers defined on noncompact domains.

**Theorem 4.1 ([14]).** *Let $X$ be a convex subset of a topological vector space and $\Phi : X \longrightarrow \mathcal{P}(X)$ be a multifunction satisfying the following properties:*

(i) $\forall y \in Y, \Phi^{-1}(y)$ *is open in* $X$;

(ii) $\forall x \in X, \Phi(x)$ *is convex nonempty;*

(iii) *there exists a compact subset $K$ of $X$ such that for any finite subset $N$ of $X$ there exists a compact convex subset $C_N$ of $X$ containing $N$ such that* $\Phi(x) \cap C_N \neq \varnothing$, $\forall x \in C_N \setminus K$.

*Then,* $\exists x_0 \in X$ *with* $x_0 \in \Phi(x_0)$.

*Proof. Step* 1. Assuming for simplicity that $X$ is compact (this correponds to the Browder–Ky Fan theorem), we can take $K = \varnothing$ and $C_N = X$ for any finite subset $N$ of $X$ and show that $\Phi$ has a fixed point as in [15]. First observe that the multifunction $\Phi$ has a so-called *Kuratowski selection*, that is: there exist a subset $N = \{y_1, ..., y_n\} \subset X$ and a single-valued continuous mapping $s : X \to \mathrm{co}(N)$ such that $s(x) \in \Phi(x)$, $\forall x \in X$. Indeed, observe that the family $\{\Phi^{-1}(y) : y \in X\}$ forms an open cover of $X$. Since $X$ is compact, there exists a finite subset $N = \{y_1, ..., y_n\}$ of $X$ such that $X = \bigcup_{i=1}^n \Phi^{-1}(y_i)$. Let $\{\lambda_i : i = 1, ..., n\}$ be a continuous partition of unity subordinated to the open cover $\{\Phi^{-1}(y_i); i = 1, ..., n\}$ of $X$. Define the continuous mapping $s : X \longrightarrow \mathrm{co}(N)$ by putting

$$s(x) := \sum_{i=1}^{n} \lambda_i(x) y_i, \ x \in X.$$

Let $x \in X$ be arbitrary. If $\lambda_i(x) \neq 0$, then $x \in \Phi^{-1}(y_i)$, hence $y_i \in \Phi(x)$. Since $\Phi(x)$ is convex, $s(x) \in \Phi(x)$. (Note here that the paracompactness of $X$ is sufficient for the existence of a continuous selection for $\Phi$; without compactness, however, the range of this selection is not necessarily finite-dimensional.)

By the Brouwer fixed point theorem, the mapping $s$ restricted to $\mathrm{co}(N)$, $s : \mathrm{co}(N) \longrightarrow \mathrm{co}(N)$ has a fixed point which is also a fixed point for $\Phi$.

*Step* 2. Now by the same argument above, the restriction of $\Phi$ to the compact set $K$ has a Kuratowski selection $s$ with values in a convex polytope $\mathrm{co}(N)$ where $N$ is a finite subset of $X$. Let $C_N$ be the compact convex subset of $X$ provided by (iii) and containing the convex hull $\text{co}(N)$ of $N$. Define the "compression" of $\Phi$ to $C_N$, $\Phi_N : C_N \longrightarrow \mathcal{P}(C_N)$, as follows:

$$\Phi_N(x) := \Phi(x) \cap C_N, x \in C_N.$$

Let us first observe that $\Phi_N$ has nonempty values. For, if $x \in C_N \cap K$, then $s(x) \in \Phi(x) \cap \text{co}(N) \subset \Phi(x) \cap C_N$; and if $x \in C_N \setminus K$, then $\Phi_N(x) \neq \varnothing$ by (iii). It is clear that $\Phi_N$ has convex values and open fibers. Hence, it has a fixed point by Step 1. This fixed point is also a fixed point for $\Phi$. □

It is well known that the celebrated inf-sup inequality of Ky Fan [2] is an equivalent analytical formulation of the Browder–Ky Fan fixed point theorem. The systematic formulation of coincidence and fixed point theorems as nonlinear alternatives was presented in Ben-El-Mechaiekh, Deguire, and Granas [14] and in [16]. Theorem 4.1 has the following convenient analytical formulation.

**Proposition 4.2.** *Let $X$ be a convex subset in a topological vector space, and $f : X \times X \longrightarrow R \cup \{+\infty\}$ be a function satisfying the following properties:*

(i) *$\forall y \in X, x \longmapsto f(x, y)$ is lower semicontinuous on $X$;*

(ii) *$\forall x \in X$, $y \longmapsto f(x, y)$ is quasiconcave on $X$.*

*Assume that there exists a compact subset $K$ of $X$ such that for every finite subset $N$ of $X$ there exists a compact convex subset $C_N$ of $X$ such that:*

(iv) *$\forall x \in C_N \setminus K$, $\exists y \in C_N \cap X$ with $f(x, y) > 0$.*

*Then one of the following properties holds:*

(a) *$\exists x_0 \in K$ such that $f(x_0, y) \leq 0$, $\forall y \in X$; or*

(b) *$\exists y_0 \in X$ such that $f(y_0, y_0) > 0$.*

*Proof.* Apply Theorem 4.1 to the multifunction $\Phi : X \longrightarrow \mathcal{P}(X)$ defined as

$$\Phi(x) := \{y \in X : f(x, y) > 0\}, x \in X. \quad \square$$

The next result is a more general version of a remarkable theorem of Ky Fan [12] and contains results of [17], [6], [18], and [8].

**Theorem 4.3.** *Let $X$ be a convex subset in a topological vector space, $Y$ a subset in $\{\varphi : X \longrightarrow R; \varphi$ is upper semicontinuous and quasiconcave$\}$, and $\Psi : X \longrightarrow \mathcal{P}(Y)$ be a multifunction. Assume that the following properties are satisfied:*

(i) *$\Psi$ admits a continuous selection $s$;*

(ii) *there exists a compact subset $K$ of $X$ such that for each finite subset $N$ of $X$ there exists a compact convex subset $C_N$ of $X$ containing $N$ such that*

$$\forall x \in C_N \setminus K, \quad \forall \varphi \in \Psi(x), \quad \varphi(x) < \max_{u \in C_N} \varphi(u).$$

*Then,*

$$\exists x_0 \in K, \exists \varphi_0 \in \Psi(x_0) \text{ such that } \varphi_0(x_0) = \max_{u \in X} \varphi_0(u).$$

*Proof.* Define $f : X \times X \longrightarrow R \cup \{+\infty\}$ by putting

$$f(x, y) := s(x)(y) - s(x)(x), \quad (x, y) \in X \times X.$$

The function $f$ satisfies hypotheses (i)–(ii) of Proposition 4.2. Moreover, given any finite subset $N$ of $X$, let $x \in C_N \setminus K$ and choose an $y \in C_N$ satisfying $s(x)(y) = \max_{u \in C_N} s(x)(u)$. By (ii), $s(x)(x) < s(x)(y)$, that is, $f(x, y) > 0$; hypothesis (iii) of Proposition 4.2 is thus satisfied. Since $f(y, y) = 0, \forall y \in X$, it follows that property (a) of Proposition 4.2 holds, that is, $f(x_0, y) \le 0$ for some $x_0 \in X$ and all $y \in X$. The proof is complete with $\varphi_0 = s(x_0)$. □

**Corollary 4.4.** *Let $X$ be a convex subset of a normed space $E$, $F$ be a normed space, $\Lambda : X \longrightarrow \Lambda(E, F)$ be a continuous mapping, and $f : X \times F^* \longrightarrow R \cup \{+\infty\}$ be two real functions satisfying the following conditions:*

(i) *$\exists M > 0$ such that $\forall x \in X$, $\forall u \in E$ with $\|u\| = 1$, $\exists v \in \Lambda(x)(u)$ such that $\|v\| \le M$;*

(ii) *$\forall \psi \in F^*$, $x \longmapsto f(x, \psi)$ is upper semicontinuous on $X$;*

(iii) *$\forall x \in X$, $\psi \longmapsto f(x, \psi)$ is quasiconvex on $F^*$.*

*Assume that there exists a compact subset $K$ of $X$ such that for each finite subset $N$ of $X$ there exists a compact convex subset $C_N$ of $X$ containing $N$ such that*

(iv) *$\forall x \in C_N \setminus K, \forall \psi \in F^*, f(x, \psi) \ge 0$ provided that $\Lambda(x)^*(\psi) \cap N_{C_N}(x) \ne \varnothing$.*

*Then one of the following conditions is satisfied:*

(1) *$\exists \widehat{x} \in X$ such that $f(\widehat{x}, \psi) \ge 0$, $\forall \psi \in F^*$; or*

(2) *$\exists (x_0, \psi_0) \in K \times F^*$ such that $\Lambda(x_0)^*(\psi_0) \cap N_X(x_0) \ne \varnothing$ and $f(x_0, \psi_0) < 0$.*

*Proof.* Note first that being a subset of a normed space, $X$ is paracompact. Assume that conclusion (1) fails and define the multifunction $\Theta : X \longrightarrow \mathcal{P}(F^*)$ with domain $X$ by putting

$$\Theta(x) := \{\psi \in F^* : f(x, \psi) < 0\}, x \in X.$$

We claim that the multifunction $\Psi : X \longrightarrow \mathcal{P}(E^*)$ defined as

$$\Psi(x) := \Lambda(x)^*(\Theta(x)), x \in X,$$

admits a continuous selection.

The multifunction $\Psi$ can be viewed as the composition product $X \xrightarrow{1_X \times \Theta} X \times \mathcal{P}(F^*) \xrightarrow{\Omega} \mathcal{P}(E^*)$, where $(1_X \times \Theta)(x) = \{x\} \times \Theta(x)$ and $\Omega(x, \psi) = \Lambda(x)^*(\psi)$, $x \in X, \psi \in F^*$.

Hypotheses (ii)–(iii) imply that the multifunction $\Theta$ has convex values and open fibers. Hence it admits a continuous selection $t$ (see Step 1 in the proof of Theorem 4.1). Clearly, $1_X \times t$ is a continuous selection of $1_X \times \Theta$.

Note that $\forall x \in X, \forall \psi \in F^*$, $\Omega(x, \psi)$ is closed and convex. If we show that the multifunction $\Omega$ is lower semicontinuous, since $X$ is paracompact and $\mathcal{P}(E^*)$ is a Banach space, then Michael's selection theorem would imply the existence of a continuous selection $s$ of the multifunction $\Omega(1_X \times t)$. This selection $s$ would clearly be a selection of $\Psi$.

In order to show that $\Omega$ is lower semicontinuous, according to Lemma 1.4 it suffices to show that the mapping $x \longmapsto \Lambda(x)^*$ is pointwise bounded and that the multifunction $x \longmapsto \operatorname{graph}(\Lambda(x)^*)$ is lower semicontinuous.

Note first that the definition (5) of the transpose of a closed convex process implies that

$$\forall x \in X, \forall \psi \in F^*, \sup_{\xi \in \Lambda(x)^*(\psi)} \|\xi\| \le \|\Lambda(x)\| \, \|\psi\|. \tag{12}$$

By (i), $\|\Lambda(x)\| = \sup_{u \in B_E(0,1)} \inf_{v \in \Lambda(x)(u)} \|v\| \le M, x \in X$; hence, (12) implies that $x \longmapsto \Lambda(x)^*$ is pointwise bounded.

In order to prove now that $x \longmapsto \operatorname{graph}(\Lambda(x)^*)$ is lower semicontinuous, let us first note that

$$\forall x, x' \in X, (\Lambda(x) - \Lambda(x'))^* = \Lambda(x)^* - \Lambda(x')^*. \tag{13}$$

Let $x \in X$ and $\epsilon > 0$ be arbitrary but fixed and let $(\psi, \varphi) \in \operatorname{graph}(\Lambda(x)^*)$. By continuity of $x \longmapsto \Lambda(x), \exists \delta > 0$ such that

$$\|\Lambda(x) - \Lambda(x')\| < \frac{\epsilon}{\|\psi\|}, \forall x' \in B_X(x, \delta).$$

By (13), given any $\varphi' \in \Lambda(x')^*(\psi)$, the linear functional $\varphi - \varphi'$ belongs to $(\Lambda(x)^* - \Lambda(x')^*)(\psi) = (\Lambda(x) - \Lambda(x'))^*(\psi)$. Hence,

$$\|\varphi - \varphi'\| \le \sup_{\xi \in (\Lambda(x)^* - \Lambda(x')^*)(\psi)} \|\xi\| \le \|\Lambda(x) - \Lambda(x')\| \, \|\psi\| < \epsilon.$$

Thus, $(\psi, \varphi) \in B_{F^* \times E^*}(\operatorname{graph}(\Lambda(x')^*), \epsilon)$, $\forall x' \in B_X(x, \delta)$. We have proved that, given any $x \in X$, the following containment is verified:

$$\begin{aligned} \operatorname{graph}(\Lambda(x)^*) \subset \bigcap_{\epsilon > 0} \bigcup_{\delta > 0} \bigcap_{x' \in B_X(x, \delta)} & B_{F^* \times E^*}(\operatorname{graph}(\Lambda(x')^*), \epsilon) = \\ &= \liminf_{x' \to x} \operatorname{graph}(\Lambda(x')^*), \end{aligned}$$

that is the multifunction $x \longmapsto \operatorname{graph}(\Lambda(x)^*)$ is lower semicontinuous.

All hypothesis of the Theorem 4.3 with $Y = F^*$ are now satisfied; hence, $\exists x_0 \in K, \exists \varphi_0 \in \Psi(x_0) = \Lambda(x_0)^*(\psi_0)$ for some $\psi_0 \in \Theta(x_0)$, with $\varphi_0 \in N_X(x_0)$. □

*Remark.* If $X$ is compact, hypothesis (i) directly follows from the continuity of $\Lambda$.

We end this section with the proof of our main theorem.

*Proof of Theorem* 3.1. (A) We apply Corollary 4.4 to $f(x,\psi) = \sigma(\Phi(x),\psi)$. Since $\Phi$ satisfies the condition of Ky Fan on $(K,X)$ with respect to $\Lambda$, Lemma 2.2 implies that $\sigma(\Phi(x),\psi) \geq 0$ for all $x \in K$ and all $\psi \in F^*$ such that $\Lambda(x)^*(\psi) \cap N_X(x) \neq \varnothing$. Thus conclusion (2) of Corollary 4.4 fails. Therefore, there exists $\widehat{x} \in X$ such that $\sigma(\Phi(\widehat{x}),\psi) \geq 0$ for all $\psi \in F^*$. The fact that $\widehat{x}$ is an equilbrium for $\Phi$ follows from the characterization (4) of the closed convex hull in terms of the support function.

(B) Given any $x_0 \in X$, define the multifunction $\Psi : X \to \mathcal{P}(F)$ by putting

$$\Psi(x) := \Phi(x) + \Lambda(x)(x - x_0), x \in X.$$

The multifunction $\Psi$ is upper hemicontinuous with closed convex values. Given any finite subset $N$ of $X$, $\Psi$ satisfies the condition of Ky Fan on $(C_N \setminus K, \mathrm{co}(\{x_0\} \cup C_N))$ and on $(K,X)$ with respect to $\Lambda$.

Let us prove that for any $x \in C_N \setminus K, \overline{\Lambda(x)(-S_{\mathrm{co}(\{x_0\}\cup C_N)}(x)} \cap \Psi(x) \neq \varnothing$. By hypothesis, there exists a net $\{y_i\}_{i\in I}$ in $\Lambda(-S_{C_N}(x))$ converging to some $y \in \Phi(x)$. For each $i \in I, y_i \in \Lambda(x)(-\frac{c_i - x}{t_i})$ for some $c_i \in C_N, t_i > 0$. Let $y'$ be any element in $\Lambda(x)(x - x_0)$. For each $i \in I$, the following containments are satisfied:

$$\begin{aligned} y_i + y' &\in \Lambda(x)\Big(-\frac{c_i - x}{t_i}\Big) + \Lambda(x)(x - x_0) \subseteq \\ &\subseteq \Lambda(x)\Big(-\Big(\frac{c_i - x}{t_i} - x + x_0\Big)\Big) = \\ &= \Lambda(x)\Big(-\Big(\frac{1+t_i}{t_i}\Big[\frac{c_i}{1+t_i} + \frac{t_i}{1+t_i}x_0 - x\Big]\Big)\Big) \subseteq \\ &\subseteq \Lambda(x)\big(-S_{\mathrm{co}(\{x_0\}\cup C_N)}(x)\big). \end{aligned}$$

Thus $y + y' \in \overline{\Lambda(x)(-S_{\mathrm{co}(\{x_0\}\cup C_N)}(x)} \cap \Psi(x)$. The proof of the fact that $\Psi$ satisfies the condition of Ky Fan on $(K,X)$ with respect to $\Lambda$ is similar.

The conclusion follows from part (A) applied to the multifunction $\Psi$. □

## 5. Some Related Results

An immediate consequence of Corollary 4.4 is the following coincidence property that generalizes a result of Ky Fan [12]:

**Proposition 5.1.** *Let $X$ be a convex subset of a normed space $E, F$ be a normed space, $\Phi, \Psi : X \longrightarrow \mathcal{P}(F)$ be two upper hemicontinuous multifunction with closed convex values, and $\Lambda : X \longrightarrow \Lambda(E, F)$ be a continuous mapping satisfying the boundedness condition:*

(i) $\exists M > 0$ *such that* $\forall x \in X$, $\forall u \in E$ *with* $\|u\| = 1, \exists v \in \Lambda(x)(u)$ *such that* $\|v\| \leq M$.

*Assume that there exist a compact subset $K$ of $X$ such that for each finite subset $N$ of $X$, there exists a compact convex subset $C_N$ of $X$ containing $N$ such that:*

(ii) $\forall x \in C_N \setminus K$, $\forall \psi \in F^*, \sigma(\Phi(x), \psi) \geq \inf_{y\in\Psi(x)}\langle \psi, y\rangle$ *provided that* $\Lambda(x)^*(\psi) \cap N_{C_N}(x) \neq \varnothing$;

(iii) $\forall x \in K$, $\forall \psi \in F^*, \sigma(\Phi(x), \psi) \geq \inf_{y\in\Psi(x)}\langle \psi, y\rangle$ *provided that* $\Lambda(x)^*(\psi) \cap N_X(x) \neq \varnothing$;

(A) *If* $\forall x \in X$ *one of the sets* $\Phi(x)$ *or* $\Psi(x)$ *is weakly compact, then* $\exists \widehat{x} \in X$ *such that* $\Phi(\widehat{x}) \cap \Psi(\hat{x}) \neq \varnothing$.

(B) *If* $\forall x \in X$ *both sets* $\Phi(x)$ *and* $\Psi(x)$ *are weakly compact, then* $\forall x_0 \in X$, $\exists \widehat{x} \in X$ *such that* $\Psi(\widehat{x}) \cap [\Phi(\hat{x}) + \Lambda(\hat{x})(\hat{x} - x_0)] \neq \varnothing$.

*Proof.* We only prove (A), the proof of (B) being an immediate consequence of Theorem 3.1 (B). Since $\forall x \in X$, one of the sets $\Phi(x)$ or $\Psi(x)$ is weakly compact, then the set $\Phi(x) - \Psi(x)$ is closed and convex. Moreover, $\inf_{y\in\Psi(x)}\langle \psi, y\rangle = -\sigma(-\Psi(x), \psi)$ and by (3), $\sigma(\Phi(x), \psi) + \sigma(-\Psi(x), \psi) = \sigma(\Phi(x) - \Psi(x), \psi)$, $\forall x \in X$, $\forall \psi \in F^*$. Corollary 4.4 applied to the function $f(x, \psi) = \sigma(\Phi(x) - \Psi(x), \psi)$ implies that the multifunction $\Phi - \Psi$ has an equilibrium which is clearly a coincidence for $\Phi$ and $\Psi$. $\square$

*Remarks.* (**1**) According to Lemma 2.2, sufficient conditions for (ii) and (iii) to hold true are:

(ii)$'$ $\Phi - \Psi$ satisfies the condition of Ky Fan on $(C_N \setminus K, C_N)$ with respect to $\Lambda$, that is,

$$[\Psi(x) + \overline{\Lambda(x)(-S_{C_N}(x))}] \cap \Phi(x) \neq \varnothing, \ \forall x \in C_N \setminus K;$$

(iii)$'$ $\Phi - \Psi$ satisfies the condition of Ky Fan on $(K, X)$ with respect to $\Lambda$, that is,

$$[\Psi(x) + \overline{\Lambda(x)(-S_X(x))}] \cap \Phi(x) \neq \varnothing, \ \forall x \in K.$$

(**2**) Again, Proposition 5.1 is true in topological spaces having sufficiently many linear functionals. It is a refinement (with tangency conditions involving a parametrized family of convex processes) of a coincidence theorem of Ky Fan [12]. Ky Fan's result corresponds to the case where $E = F$, $\Lambda(x) = Id_E$ for all $x, C_N = K$ for all $N$.

(**3**) When $\Lambda(x) \equiv \ell$ is a bounded linear operator, this result can be found in [6]. Particular forms of this result can be found in [3], [4], [7], [8] and others.

In the case where $E = F$ and $\Psi$ is the inclusion $X \hookrightarrow E$, we obtain a fixed point theorem for inward or outward multifunctions that generalizes Ky Fan fixed point theorem. By way of illustration, we state this fixed point property in the case where $X$ is compact. We will say that the multifunction $\Phi : X \longrightarrow \mathcal{P}(E)$ is *inward* with respect to a continuous family of closed convex processes $\Lambda : X \longrightarrow \Lambda(E, E)$ if the following property is verified:

$$\Phi(x) \cap [x - \overline{\Lambda(x)(-S_X(x))}] \neq \varnothing, \quad \forall x \in X.$$

The multifunction $\Phi$ is said to be *outward* with respect to $\Lambda$ if

$$\Phi(x) \cap [x + \overline{\Lambda(x)(-S_X(x))}] \neq \varnothing, \ \forall x \in X.$$

**Corollary 5.2.** *Assume that $X$ is a compact convex subset of a normed space $E$ and that the multifunction $\Phi : X \longrightarrow \mathcal{P}(E)$ is upper hemicontinuous with nonempty closed convex values. If $\Phi$ is inward or outward with respect to $\Lambda$, then it has a fixed point.*

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