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Λ_0-NUCLEAR OPERATORS AND Λ_0-NUCLEAR SPACES IN p-ADIC ANALYSIS

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Abstract. For a Köthe sequence space, the classes of Λ_0-nuclear spaces and spaces with the Λ_0-property are introduced and studied and the relation between them is investigated. Also, we show that, for $\Lambda_0 \neq c_0$, these classes of spaces are in general different from the corresponding ones for $\Lambda_0 = c_0$, which have been extensively studied in the non-archimedean literature (see, for example, [1]–[6]).

Introduction

Throughout this paper K will be a complete non-archimedean valued field whose valuation $|\cdot|$ is non-trivial, and $E, F, \dots$ will be locally convex spaces over K. We always assume that $E, F, \dots$ are Hausdorff.

It is well known (see [5]) that a locally convex space E is nuclear if and only if

(1) For every Banach space F, every continuous linear map (or operator) from E into F is compact.

Nuclear spaces are closely related to the locally convex spaces E satisfying the following property:

(2) Every operator from E into c_0 is compact (see [5]).

On the other hand, it is well known that if F is a normed space, then an operator T from E into F is compact if and only if there exist an equicontinuous sequence (f_n) in E', a bounded sequence (y_n) in F, and an element (λ_n) of c_0 such that

$$T(x) = \sum_{n=1}^{\infty} \lambda_n f_n(x) y_n \quad \forall x \in E \qquad (*)$$

(an operator satisfying this condition is called a nuclear operator, see [7]).

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In [7] and [8] the authors studied several properties of operators T which can be represented as in (*) where (λ_n) belongs to some Köthe sequence space Λ_0. They are called Λ_0-nuclear operators.

Let us introduce for a locally convex space E the following properties:

(1′) For every Banach space F, every operator from E into F is Λ_0-nuclear.

(2′) Every operator from E into c_0 is Λ_0-nuclear.

In this paper we study property (1′) as related to (2′). We show that if $\Lambda_0 \neq c_0$, the class of spaces satisfying property (1′) (resp. (2′)) is in general different from the corresponding one for $\Lambda_0 = c_0$.

In the classical case of spaces over the real or complex field, analogous problems have been studied by several authors (see, for example, [9]–[14]).

§ 1. Preliminaries

Let E be a locally convex space over K. We will denote by $\mathrm{cs}(E)$ the collection of all continuous non-archimedean seminorms on E. For $p \in \mathrm{cs}(E)$, E_p will be the associated normed space $E/\ker p$ endowed with the usual norm, and $\pi_p : E \longrightarrow E_p$ will be the canonical surjection. E is said to be of countable type if for every $p \in \mathrm{cs}(E)$, E_p is a normed space of countable type (i.e., E_p is the closed linear hull of a countable set). For $p \in \mathrm{cs}(E)$ and $r > 0$, $B_p(0, r)$ will be the set $\{x \in E : p(x) \leq r\}$. Also, for each continuous linear functional $f \in E'$, we define $\|f\|_p = \sup\{|f(x)|/p(x) : x \in E,\ p(x) \neq 0\}$.

Next, we will recall the definition of a non-archimedean Köthe space $\Lambda(P)$. By a Köthe set we will mean a collection P of sequences $\alpha = (\alpha_n)$ of non-negative real numbers with the following properties:

(1) For each $n \in N$ there exists $\alpha \in P$ with $\alpha_n \neq 0$.

(2) If $\alpha, \alpha' \in P$, then there exists $\beta \in P$ with $\alpha, \alpha' \ll \beta$, where $\alpha \ll \beta$ means that there exists $d > 0$ such that $\alpha_n \leq d\beta_n$ for all n.

For $\alpha \in P$ and a sequence $\xi = (\xi_n)$ in K, we define $p_\alpha(\xi) = \sup_n \alpha_n |\xi_n|$. The non-archimedean Köthe sequence space $\Lambda(P)$ is the space of all sequences ξ in K for which $p_\alpha(\xi) < \infty$ for all $\alpha \in P$. On $\Lambda(P)$ we consider the locally convex topology generated by the family $\{p_\alpha : \alpha \in P\}$ of non-archimedean seminorms. Under this topology $\Lambda(P)$ is a complete Hausdorff locally convex space over K. The set $|\Lambda| = \{|x| : x \in \Lambda(P)\}$ is a Köthe set. By $\bar{\Lambda}$ we will denote the Köthe space $\Lambda(|\Lambda|)$. Also, by $\Lambda_0 = \Lambda_0(P)$ we will denote the closed subspace of $\Lambda(P)$ consisting of all $\xi = (\xi_n)$ for which $\alpha_n|\xi_n| \to 0$ for each $\alpha = (\alpha_n) \in P$. In case P consists of a single constant sequence $(1, 1, \dots)$, we have $\Lambda(P) = \ell^\infty$ and $\Lambda_0(P) = c_0$. Also, we give the following interesting example:

Let $B = (b_n^k)$ be an infinite matrix of strictly positive real numbers and satisfying the conditions $b_n^k \leq b_n^{k+1}$ for all k, n. For each k, let $\alpha^{(k)} =$

$(b_1^k, b_2^k, \dots)$. Then, $P = \{\alpha^{(k)} : k = 1, 2, \dots\}$ is a Köthe set for which $\Lambda_0(P)$ coincides with the Köthe space $K(B) = \{(\lambda_n) : \lambda_n \in K, \forall n$ and $\lim_n |\lambda_n| b_n^k = 0,\ k = 1, 2, 3, \dots\}$ associated with the matrix B (see [4]). Also, the topology on $\Lambda_0(P)$ for this P coincides with the normal topology on $K(B)$ considered in [4]. This kind of spaces play an important role in p-adic analysis, since every non-archimedean countably normed Fréchet space E with a Schauder basis can be identified with $K(B)$, for some infinite matrix B ([4], Proposition 2.4).

We will say that the Köthe set P is a power set of infinite type if (i): For each $\alpha \in P$ we have $0 < \alpha_n \leq \alpha_{n+1}$ for all n, and (ii): For each $\alpha \in P$ there exists $\beta \in P$ with $\alpha^2 \ll \beta$. We will say that P is stable if for each $\alpha \in P$ there exists $\beta \in P$ such that $\sup_n \alpha_{2n}/\beta_n < \infty$. By [7], Proposition 2.11, P is stable if and only if $\Lambda(P)$ (or $\Lambda_0(P)$) is stable. (Recall that a locally convex space E is called stable if $E \times E$ is topologically isomorphic to E.)

Finally, we will recall the concepts of Λ_0-compactoid sets and Λ_0-nuclear operators (see [7]). For a bounded subset A of a locally convex space E, $p \in \mathrm{cs}(E)$ and a non-negative integer n, the nth Kolmogorov diameter $\delta_{n,p}(A)$ of A with respect to p is the infimum of all $|\mu|$, $\mu \in K$, for which there exists a subspace F of E with $\dim(F) \leq n$ such that $A \subset F + \mu B_p(0,1)$. The set A is called Λ_0-compactoid if for each $p \in \mathrm{cs}(E)$ there exists $\xi = (\xi_n) \in \Lambda_0$ such that $\delta_{n,p}(A) \leq |\xi_{n+1}|$ for all n (or equivalently $\alpha_n \delta_{n-1,p}(A) \to 0$ for all $\alpha \in P$). An operator (continuous linear map) $T \in L(E, F)$ between two locally convex spaces E,F over K is called:

(1) Λ_0-nuclear if there exist an equicontinuous sequence (f_n) in E', a bounded sequence (y_n) in F, and an element (λ_n) of Λ_0 such that

$$Tx = \sum_{n=1}^{\infty} \lambda_n f_n(x) y_n \quad \forall x \in E;$$

(2) Λ_0-compactoid if there exists a neighborhood V of zero in E such that $T(V)$ is Λ_0-compactoid in F;

(3) Λ_0-quasinuclear if for each $q \in \mathrm{cs}(F)$ there exist a sequence (f_n) in E', a $p \in \mathrm{cs}(E)$, and an element (λ_n) of Λ_0 such that $\|f_n\|_p \leq |\lambda_n|$ $(n \in N)$ and $q(Tx) \leq \sup_n |f_n(x)|$ for all $x \in E$. (For the ideal structure of these classes of operators see [7].)

By Theorem 4.4 of [7], every Λ_0-nuclear operator is Λ_0-compactoid. Also, every Λ_0-compactoid operator is Λ_0-quasinuclear. Indeed, if T is Λ_0-compactoid and $q \in \mathrm{cs}(F)$, then $\pi_q \circ T : E \longrightarrow F_q$ is also Λ_0-compactoid ([7], Proposition 3.21) and so $\pi_q \circ T$ is Λ_0-nuclear ([7], Theorem 4.7). Hence, T is Λ_0-quasinuclear.

It follows from Theorem 4.6 of [7] that if F is a normed space, then T is Λ_0-nuclear $\Leftrightarrow$ T is Λ_0-compactoid $\Leftrightarrow$ T is Λ_0-quasinuclear.

In case $\Lambda_0 = c_0$, the concepts of Λ_0-compactoid set, Λ_0-compactoid operator, and Λ_0-nuclear operator coincide with the concepts of a compactoid set, a compact operator, and a nuclear operator, respectively.

For further information we refer to [15] (for normed spaces) and to [16] (for locally convex spaces).

From now on in this paper we will assume that the Köthe set P is a power set of infinite type.

§ 2. Spaces with the Λ_0-Property

Locally convex spaces E for which every $T \in L(E, c_0)$ is compact have been studied by N. De Grande-De Kimpe in [2] and [3] and more recently by T. Kiyosawa in [6].

A natural extension of this kind of spaces is given by

Definition 2.1. We say that a locally convex space E has the Λ_0-property if every $T \in L(E, c_0)$ is Λ_0-nuclear (or, equivalently, Λ_0-compactoid).

In this section, we study several properties of spaces with the Λ_0-property. In this way, we extend and complete the results previously obtained by N. De Grande-De Kimpe and T. Kiyosawa.

Proposition 2.2.

(a) *If E has the Λ_0-property and M is a subspace of E such that every $T \in L(M, c_0)$ has an extension $\bar{T} \in L(E, c_0)$ (e.g., when M is dense or when M is complemented), then M has the Λ_0-property.*

(b) *A locally convex space E has the Λ_0-property if and only if its completion $\hat{E}$ has the Λ_0-property.*

(c) *A quotient of a space E with the Λ_0-property also has the same property.*

(d) *If P is stable, then the product of a family of spaces with the Λ_0-property has the same property.*

Proof. Property (a) is obvious.

(b): It follows by (a) that if $\hat{E}$ has the Λ_0-property, then E has also the same property.

Conversely, suppose that E has the Λ_0-property. Let $T \in L(\hat{E}, c_0)$ and let T_1 be the restriction of T to E. Since T_1 is Λ_0-compactoid, there exists a zero-neighborhood U in E such that $T_1(U)$ is Λ_0-compactoid in c_0. Then $V = \bar{U}^{\hat{E}}$ is a zero-neighborhood in $\hat{E}$ for which $T(V)$ is Λ_0-compactoid in c_0, and so T is Λ_0-compactoid.

(c): Let M be a closed subspace of E and let $S \in L(E/M, c_0)$. If $\pi : E \longrightarrow E/M$ is the quotient map, then $T = S \circ \pi \in L(E, c_0)$ is Λ_0-compactoid. If V is a neighborhood of zero in E such that $T(V)$ is Λ_0-compactoid in c_0, then $\pi(V)$ is a neighborhood of zero in E/M for which $S(\pi(V)) = T(V)$ is Λ_0-compactoid in c_0. Hence S is Λ_0-compactoid.

(d): Let $E = \prod_i E_i$, where each E_i has the Λ_0-property, and let $T \in L(E, c_0)$. Then T is bounded on a neighborhood W of zero in E. This neighborhood can be taken in the form $W = \prod_i U_i$, where U_i is a zero-neighborhood in E_i and the set $J = \{i \in I : U_i \neq E_i\}$ is finite. Clearly, T vanishes on the subspace $\prod_{i \notin J} E_i$ of E and so we may assume that I is finite, i.e., $E = E_1 \times E_2 \times \ldots \times E_n$ for some $n \in N$. For $j = 1, 2, ..., n$, let $\pi_j : E_j \longrightarrow E$ be the canonical inclusion. Since $T_j = T \circ \pi_j \in L(E_j, c_0)$ is Λ_0-compactoid, there exists a zero-neighborhood V_j in E_j such that $T_j(V_j)$ is Λ_0-compactoid in c_0. Then, $V = V_1 \times V_2 \times \cdots \times V_n$ is a zero neighborhood in E for which $T(V) = T_1(V_1) + \ldots + T_n(V_n)$ is Λ_0-compactoid in c_0 ([7], Proposition 3.14). Thus T is Λ_0-compactoid. □

Now, we fix some notation which we will use in the sequel. For each $n \in N$, there are unique $k, m \in N$ such that $n = (2m-1)2^{k-1}$. In the following lemma $\pi_1, \pi_2 : N \longrightarrow N$ will be defined by $\pi_1(n) = k$ and $\pi_2(n) = m$ when $n = (2m-1)2^{k-1}$.

Lemma 2.3. *Suppose that P is countable and stable and, for each $k \in N$, let $\xi^k = (\xi^k_n)_n \in \Lambda_0$. Then there exists a sequence $(\lambda_k)_k$ of non-zero elements of K such that $(\lambda_{\pi_1(n)}\xi^{\pi_1(n)}_{\pi_2(n)})_n \in \Lambda_0$.*

Proof. We may assume that $P = (\alpha^{(k)})_{k \in N}$, where $\alpha^{(k)} \leq \alpha^{(k+1)}$ for all k. Since P is stable, we may also assume that for each $k \in N$ there exists $0 < d_k < \infty$ with $d_k \leq d_{k+1}$ such that $\sup_m \alpha^{(k)}_{2^k.m}/\alpha^{(k+1)}_m \leq d_k$. Choose $\lambda_k \in K$, $0 < |\lambda_k| \leq 1$ such that $p_{\alpha^{(k+1)}}(\lambda_k \xi^k) \leq k^{-1} d_k^{-1}$ $(k \in N)$. We claim that the sequence $(\lambda_k)_k$ satisfies the requirements.

Indeed, let $r \in N$ and let $\epsilon > 0$ be given. Choose $k_0 > \max\{r, 1/\epsilon\}$. Also, choose $\eta^k_n \in K$ with $|\eta^k_n| = \max_{m \geq n} |\xi^k_m|$ $(k, n \in N)$. Then, $\eta^k = (\eta^k_n)_n \in \Lambda_0$ for all $k \in N$ and so there exists $m_0 \in N$ such that $d_{k_0}\alpha^{(k_0)}_m |\eta^k_m| < \epsilon$ for all $m \geq m_0$ and all $k \leq k_0$. Let $n > m_0 2^{k_0}$. If $k = \pi_1(n) < r$, then $k < k_0$ and hence $m > m_0$. Thus, for $k = \pi_1(n) < r$, we have

$$\alpha^{(r)}_n |\lambda_{\pi_1(n)} \xi^{\pi_1(n)}_{\pi_2(n)}| \leq \alpha^{(r)}_n |\eta^k_m| \leq \alpha^{(r)}_{m2^r} |\eta^k_m| \leq d_{k_0} \alpha^{(k_0)}_m |\eta^k_m| < \epsilon.$$

For $r \leq k = \pi_1(n) < k_0$, we have

$$\alpha^{(r)}_n \leq \alpha^{(k)}_{m2^k} \leq d_k \alpha^{(k+1)}_m \leq d_{k_0} \alpha^{(k_0)}_m$$

and, since $m > m_0$ we obtain that

$$\alpha_n^{(r)}|\lambda_k \xi_m^k| \leq d_{k_0} \alpha_m^{(k_0)} |\xi_m^k| < \epsilon.$$

Analogously, we can prove that if $\pi_1(n) = k \geq k_0 > r$, then we have $\alpha_n^{(r)}|\lambda_k \xi_m^k| < \epsilon$.

Hence, for $n > m_0.2^{k_0}$, we obtain $\alpha_n^{(r)}|\lambda_{\pi_1(n)} \xi_{\pi_2(n)}^{\pi_1(n)}| < \epsilon$, which clearly completes the proof. □

Theorem 2.4. *Let P be countable and stable. Then the locally convex direct sum and the inductive limit of a sequence of spaces with the Λ_0-property have also the same property.*

Proof. Let $E = \bigoplus_{k=1}^{\infty} E_k$, where each E_k has the Λ_0-property and let $T \in L(E, c_0)$. If $I_k : E_k \longrightarrow E$ is the canonical inclusion, then $T \circ I_k \in L(E_k, c_0)$ is Λ_0-nuclear ($k \in N$). Therefore, for each k, there exist $\xi^k = (\xi_m^k)_m \in \Lambda_0$, a sequence $(y_m^k)_m$ in the unit ball of c_0, and an equicontinuous sequence $(h_m^k)_m$ in E_k' such that

$$(T \circ I_k)(y) = \sum_{m=1}^{\infty} \xi_m^k h_m^k(y) y_m^k \quad (y \in E_k).$$

For each $k \in N$ let $q_k \in \mathrm{cs}(E_k)$ with $|h_m^k| \leq q_k$ for all m. Also, let π_1, π_2 and $(\lambda_k)_k$ be as in Lemma 2.3. Then $q(x) = \max_k |\lambda_k|^{-1} q_k(x_k)$ $(x = (x_k)_k \in E)$ defines a continuous seminorm on E. For each pair (m, k) of positive integers, the function $g_m^k : E \longrightarrow K$, $x \longrightarrow \lambda_k^{-1} h_m^k(x_k)$ is a continuous linear map on E such that $|g_m^k| \leq q$ for all k, m. Also, for each $x = (x_k)_k = \sum_{k=1}^{\infty} I_k(x_k) \in E$ we have

$$Tx = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \lambda_k \xi_m^k g_m^k(x) y_m^k.$$

For $n = (2m-1)2^{k-1}$, set $f_n = g_m^k \in E'$, $z_n = y_m^k \in c_0$, $\xi_n = \lambda_k \xi_m^k \in K$. By Lemma 2.3 $(\xi_n)_n \in \Lambda_0$. Further, $Tx = \sum_{n=1}^{\infty} \xi_n f_n(x) z_n$ for all $x \in E$, and so T is Λ_0-nuclear.

Finally, we observe that the inductive limit of a sequence of spaces is linearly homeomorphic to a quotient of the corresponding direct sum. □

Remark. A subspace of a space with the Λ_0-property need not have in general the same property. Indeed, let $\Lambda_0 = c_0$ and suppose that the valuation on K is dense. Then, ℓ^∞ has the Λ_0-property ([15], Corollary 5.19) but, clearly, c_0 does not have the same property.

Examples.

1. As we will see in the next section, every Λ_0-nuclear space has the Λ_0-property.

2. If $\Lambda_0 = c_0$ and the valuation on K is dense, then ℓ^∞ has the Λ_0-property.

3. If P is countable and K is not spherically complete, then $\bar{\Lambda}$ has the Λ_0-property ([8], Corollary 4.6).

4. If E is an infinite-dimensional Banach space with a basis, then E does not have the Λ_0-property. Indeed, E contains a complemented subspace linearly homeomorphic to c_0 ([15], Corollary 3.18).

For a locally convex space E over K, we will denote by $\Lambda_0\{E'\}$ the family of all sequences (g_n) in E' for which there exist $p \in \text{cs}(E)$ and $(\lambda_n) \in \Lambda_0$ such that $\|g_n\|_p \leq |\lambda_n|$ for all n. For a sequence $w = (g_n) \in \Lambda_0\{E'\}$, we define a continuous non-archimedean seminorm p_w on E by

$$p_w(x) = \sup_n |g_n(x)| \qquad (x \in E).$$

The next Theorem gives several descriptions of spaces with the Λ_0-property.

Theorem 2.5. *For a locally convex space E, the following properties are equivalent:*

(i) *E has the Λ_0-property.*

(ii) *For every $T \in L(E, c_0)$ there exist $T_1 \in L(E, \Lambda_0)$, which is Λ_0-nuclear, and $T_2 \in L(\Lambda_0, c_0)$ such that $T = T_2 \circ T_1$.*

(iii) *If F is a locally convex space of countable type, then every $T \in L(E, F)$ is Λ_0-quasinuclear.*

(iv) *If F is a normed space and $T \in L(E, F)$, then T is Λ_0-nuclear if and only if its range, $R(T)$, is of countable type.*

(v) *Let (T_n) be an equicontinuous sequence of operators from E into a normed space F such that $R(T_n)$ is of countable type for all n and such that (T_n) converges pointwise to a $T \in L(E, F)$. Then T is Λ_0-nuclear.*

(vi) *For every equicontinuous sequence (f_n) in E', which converges pointwise to zero, there exists $w \in \Lambda_0\{E'\}$ such that $\|f_n\|_{p_w} \leq 1$ for all n.*

(vii) *For every equicontinuous sequence (f_n) in E', which converges pointwise to zero, there exist $(g_n) \in \Lambda_0\{E'\}$, $\alpha \in P$, $d > 0$, and an infinite matrix (ξ_{ik}) of elements of K, with $\lim_{n\to\infty} \xi_{in} = 0$ for all i and $|\xi_{in}| < d\alpha_i$ for all n, such that*

$$f_n(x) = \sum_{i=1}^{\infty} g_i(x)\xi_{in} \quad (x \in E).$$

If, in addition, P is stable, then properties (i) $\rightarrow$ (vii) *are equivalent to:*

(viii) *The topology of uniform convergence on the members of $\Lambda_0\{E'\}$ coincides with the topology τ_0 of countable type which is associated with the topology of E (see* [17]*).*

Proof. For the equivalence of (i) and (ii) see the proof of Theorem 4.6 in [7].

(i) $\Rightarrow$ (iii): Let F be a locally convex space of countable type. For every $p \in \mathrm{cs}(F)$, the associated normed space F_p is of countable type and so F_p is linearly homeomorphic to a subspace of c_0. Hence $\pi_p \circ T : E \longrightarrow F_p$ is Λ_0-nuclear ([7], Theorem 4.11). Thus T is Λ_0-quasinuclear.

(iii) $\Rightarrow$ (iv): Observe that, since $\Lambda_0 \subset c_0$, we have that every Λ_0-nuclear operator is also nuclear, and hence its range is of countable type.

(iv) $\Rightarrow$ (v): Let (T_n) and T be as in (v). Since every $R(T_n)$ is of countable type, the closed linear hull Z of $\bigcup_n R(T_n)$ is of countable type. Also, since $Tx \in Z$ for all $x \in E$, (iv) implies that T is Λ_0-nuclear.

(i) $\Leftrightarrow$ (vi): From Theorem 4.6 of [7] it follows that a map $T \in L(E, c_0)$ is Λ_0-nuclear if and only if there exists $w \in \Lambda_0\{E'\}$ such that $\|Tx\| \le p_w(x)$ for all $x \in E$. Now, apply Lemma 2.2 of [3] to get the conclusion.

(ii) $\Leftrightarrow$ (vii): By Lemma 2.2 of [3] it follows that a linear map T from Λ_0 into c_0 is continuous if and only if there exist an infinite matrix (ξ_{ij}) of elements of K, an $\alpha \in P$ and $d > 0$ such that $|\xi_{ij}| \le d\alpha_i$ for all i, j, $\lim_{j\to\infty} \xi_{ij} = 0$ for all i and $Tx = (\sum_{i=1}^{\infty} x_i \xi_{ij})_j$ for all $x = (x_i) \in \Lambda_0$. Also, by Theorem 3.3 of [8], it follows that a linear map $S \in L(E, \Lambda_0)$ is Λ_0-nuclear if and only if there exists $(g_n) \in \Lambda_0\{E'\}$ such that $Tx = (g_n(x))_n$ for all $x \in E$. Now, the conclusion follows again by Lemma 2.2 of [3].

Finally, suppose that P is stable.

(vi) $\Leftrightarrow$ (viii): We first observe that, since P is stable, the family of seminorms $\{p_w : w \in \Lambda_0\{E'\}\}$ is upwards directed. Also, we know that τ_0 is the topology of uniform convergence on the equicontinuous sequences in E' which converge pointwise to zero. Now, the result follows. □

Remark. If a locally convex space E has the Λ_0-property, then every $T \in L(E, c_0)$ is compact, since $\Lambda_0 \subset c_0$. But the converse is not true in general.

Example. Suppose that the valuation on K is dense. It is well known that every $T \in L(\ell^\infty, c_0)$ is compact. However, if $\Lambda_0 \neq c_0$, there are operators from ℓ^∞ to c_0 which are not Λ_0-nuclear ([8], Corollary 3.7).

§ 3. Λ_0-Nuclear Spaces

Nuclear spaces have been extensively studied in the non-archimedean literature (see, for example, [5] for a collection of the basic properties of these spaces). A natural extension of this kind of spaces is the following:

Definition 3.1. A locally convex space E is called Λ_0-nuclear if for each $p \in \mathrm{cs}(E)$ there exists $q \in \mathrm{cs}(E)$, $p \le q$, such that the canonical map $\Phi_{pq} : E_q \longrightarrow E_p$ is Λ_0-nuclear (or, equivalently, Λ_0-compactoid).

In this section we study the relationship between the Λ_0-nuclear spaces and the spaces with the Λ_0-property considered in the previous section. We first need some preliminary machinery.

Let $m \in N$ and let $\xi^{(1)}, \dots, \xi^{(m)}$ be m elements of Λ_0. For $j = (n-1)m + k$, where $1 \le k \le m$, set $\xi_j = \xi_n^{(k)}$. If P is stable, then $\xi = (\xi_j) \in \Lambda_0$ (we will denote ξ by $\xi^{(1)} * \xi^{(2)} * \dots \xi^{(m)}$).

Indeed, let $\alpha \in P$ and let $m_1 \in N$ be such that $m \le 2^{m_1}$. Since P is stable, there exist $\beta \in P$ and $d > 0$ such that $\alpha_{n.2^{m_1}}/\beta_n \le d$ for all n. Given $\epsilon > 0$, there exists $n_0 \in N$ such that $d\beta_n|\xi_n^{(k)}| < \epsilon$ for $k = 1, \dots, m$ and $n \ge n_0$. If $j \ge n_0 m$ and $j = (n-1)m + k$, then $n \ge n_0$ and so

$$\alpha_j|\xi_j| \le \alpha_{nm}|\xi_j| \le \alpha_{n.2^{m_1}}|\xi_j| \le d\beta_n|\xi_n^{(k)}| < \epsilon.$$

Lemma 3.2. *Let P be stable. Then, for each positive integer m, the function $\Psi_m : \Lambda_0^m \longrightarrow \Lambda_0$, $\Psi_m(\xi^{(1)}, \dots, \xi^{(m)}) = \xi^{(1)} * \dots * \xi^{(m)}$ is a linear homeomorphism from Λ_0^m onto Λ_0.*

Proof. It is easy to see that Ψ_m is a bijection. To prove the continuity of Ψ_m, recall that, given $\alpha \in P$, there exist $\beta \in P$ and $d > 0$ such that $\alpha_{nm} \le d\beta_n$ for all n, and so, $p_\alpha(\Psi_m(\xi)) \le d \max_{1\le k\le m} p_\beta(\xi^{(k)})$ for all $\xi = (\xi^{(1)}, \dots, \xi^{(m)}) \in \Lambda_0^m$ which proves that Ψ_m is continuous.

Also, Ψ_m^{-1} is continuous. In fact, for $\xi = (\xi_n) \in \Lambda_0$ we have $\Psi_m^{-1}(\xi) = (\xi^{(1)}, \dots, \xi^{(m)})$, where $\xi^{(k)} = (\xi_k, \xi_{m+k}, \xi_{2m+k}, \dots)$ $(k = 1, \dots, m)$. Also, for each $\alpha \in P$ we get $p_\alpha(\xi) \ge \max_{1\le k\le m} p_\alpha(\xi^{(k)})$, and the result follows. □

Proposition 3.3. *For a locally convex space E consider the following properties:*

(i) *For every Banach space F and for every $T \in L(E, F)$, there are $T_1 \in L(E, \Lambda_0)$ and $T_2 \in L(\Lambda_0, F)$ such that $T = T_2 \circ T_1$.*

(ii) *E is of countable type and for every $T \in L(E, c_0)$ there exist $T_1 \in L(E, \Lambda_0)$ and $T_2 \in L(\Lambda_0, c_0)$ such that $T = T_2 \circ T_1$.*

(iii) *If $\{p_i : i \in I\}$ is a generating family of continuous seminorms on E, then E is linearly homeomorphic to a subspace of the product space Λ_0^I.*

(iv) *E is linearly homeomorphic to a subspace of Λ_0^J for some set J.*

Then, (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

If, in addition, P is stable, then properties (i) $\to$ (iv) *are equivalent.*

Proof. The implication (i) $\Rightarrow$ (iii) can be proved analogously to (1) $\Rightarrow$ (2) in Proposition 3.7 of [18].

(i) $\Rightarrow$ (ii): Since (i) implies (iii) and since Λ_0 is of countable type, we derive that E is also of countable type ([16], Proposition 4.12).

(ii) $\Rightarrow$ (i): Let F be a Banach space and let $T \in L(E, F)$.

First, assume that the range, $R(T)$, is finite-dimensional. Then, there exists a linear homeomorphism h from $R(T)$ onto a closed subspace M of

Λ_0. On the other hand, since the dual of Λ_0 separates the points, there exists a continuous linear projection Q from Λ_0 onto M. Hence $T = T_2 \circ T_1$, where $T_1 = h \circ T \in L(E, \Lambda_0)$ and $T_2 = h^{-1} \circ Q \in L(\Lambda_0, F)$.

Now, assume that $R(T)$ is infinite-dimensional. Since E is of countable type, the closure of $R(T)$ is an infinite-dimensional Banach space of countable type and so it is linearly homeomorphic to c_0. Now, the conclusion follows by (ii).

Now, assume that P is stable. Then, the implication (iv) $\Rightarrow$ (i) can be proved by using Lemma 3.2 in a similar way as (3) $\Rightarrow$ (1) in Proposition 3.7 of [18]. □

As in Theorems 3.2 and 3.4 of [18] we obtain the following

Proposition 3.4. *For a locally convex space E, consider the following properties:*

(i) *E is Λ_0-nuclear.*

(ii) *For every locally convex space F, every $T \in L(E, F)$ is Λ_0-quasinuclear.*

(iii) *For every Banach space F, every $T \in L(E, F)$ is Λ_0-nuclear.*

(iv) *For every $p \in \mathrm{cs}(E)$ there exists $w \in \Lambda_0\{E'\}$ such that $p \le p_w$.*

(v) *The topology of E coincides with the topology of uniform convergence on the members of $\Lambda_0\{E'\}$.*

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v).

If, in addition, P is stable, then properties (i) $\rightarrow$ (v) *are equivalent.*

It is well known (see, for example, [5], Proposition 5.4) that a locally convex space E is nuclear if and only if E is of countable type and every $T \in L(E, c_0)$ is compact. Now, using Propositions 3.3 and 3.4 we get the following descriptions of Λ_0-nuclear spaces.

Theorem 3.5. *For a locally convex space E, consider the following properties:*

(i) *E is Λ_0-nuclear.*

(ii) *For every Banach space F and every $T \in L(E, F)$, there exists $T_1 \in L(E, \Lambda_0)$ Λ_0-nuclear and $T_2 \in L(\Lambda_0, F)$ such that $T = T_2 \circ T_1$.*

(iii) *E has the Λ_0-property and it is linearly homeomorphic to a subspace of Λ_0^I for some set I.*

(iv) *E is of countable type and has the Λ_0-property.*

(v) *E is linearly homeomorphic to a subspace of some product Λ_0^I and every $T \in L(E, \Lambda_0)$ is Λ_0-quasinuclear.*

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v).

If, in addition, P is stable, then properties (i) $\rightarrow$ (v) *are equivalent.*

Proof. By using Proposition 3.4, the implication (i) $\Rightarrow$ (ii) can be proved as in Theorem 4.6 of [7].

(ii) $\Rightarrow$ (iii): It follows from Proposition 4.5 of [7] and Proposition 3.3.

(iii) $\Rightarrow$ (iv): It is obvious (recall that Λ_0 is of countable type).

(iv) $\Rightarrow$ (i): Let F be a Banach space and let $T \in L(E, F)$. Since E is of countable type, we have that the closure of $R(T)$ is a Banach space of countable type, and so it is linearly homeomorphic to a subspace of c_0. By (iv) and Theorem 4.11 of [7] we derive that T is Λ_0-nuclear. Now, the conclusion follows by Proposition 3.4.(i) $\Leftrightarrow$ (iii).

(iii) $\Rightarrow$ (v): It is a direct consequence of Theorem 2.5.(i) $\Rightarrow$(iii).

Finally, if P is stable, the implication (v) $\Rightarrow$ (iii) follows from Proposition 4.5 of [7] and our Proposition 3.3. $\square$

Putting together Proposition 2.2, Theorems 2.4, 3.5 and the stability properties of spaces of countable type ([16], Proposition 4.12), we obtain the following extension of 5.7 of [5] and Proposition 3.5 of [19].

Corollary 3.6.

(a) *Every subspace of a Λ_0-nuclear space is again Λ_0-nuclear.*

(b) *A locally convex space E is Λ_0-nuclear if and only if its completion $\hat{E}$ is Λ_0-nuclear.*

(c) *A quotient of a Λ_0-nuclear space is also Λ_0-nuclear.*

(d) *If P is stable, then the product of a family of Λ_0-nuclear spaces is also Λ_0-nuclear.*

(e) *If P is countable and stable, then the locally convex direct sum and the inductive limit of a sequence of Λ_0-nuclear spaces are also Λ_0-nuclear.*

§ 4. Some Remarks and Examples

It is well known that if E is a nuclear space, then every bounded subset of E is compactoid. The corresponding counterpart is also true for Λ_0-nuclear spaces.

Proposition 4.1. *Each bounded subset of a Λ_0-nuclear space E, is Λ_0-compactoid.*

Proof. Let B be a bounded set of E and let $p \in \mathrm{cs}(E)$. Since $\pi_p : E \longrightarrow E_p$ is Λ_0-compactoid (Proposition 3.5), we have that $\pi_p(B)$ is Λ_0-compactoid in E_p. By [7], Proposition 3.10, we derive that B is Λ_0-compactoid in E. $\square$

Remark. The converse of Proposition 4.1 is not true in general. For an example see [20].

Now, we will give some examples of spaces which are, or are not, Λ_0-nuclear.

By Proposition 3.4 and with an argument analogous to the one used in the proof of Theorem 5.2 in [18], we can obtain the following result which will be crucial for our purpose.

Theorem 4.2. *Let Q be a Köthe set (not necessarily of infinite type). Then the following properties are equivalent:*
(i) $\Lambda(Q)$ *is* $\Lambda_0(P)$*-nuclear.*
(ii) $\Lambda_0(Q)$ *is* $\Lambda_0(P)$*-nuclear.*
(iii) *For each* $\alpha \in Q$ *there exist* $\beta \in Q$ *with* $\alpha \ll \beta$*, a permutation* σ *on* N*, and* $(\lambda_n) \in \Lambda_0(P)$ *such that* $\alpha_{\sigma(n)} \leq |\lambda_n| \beta_{\sigma(n)}$ *for all* $n \in N$*.*

As a direct consequence we derive the following assertion (cf. [4], Proposition 3.5).

Corollary 4.3. *Let* $K(B)$ *be the Köthe space associated to an infinite matrix* $B = (b_n^k)$*. Then K(B) is* Λ_0*-nuclear if and only if for every* k *there exist* $k_1 > k$*, a permutation* σ *on* N*, and* $(\lambda_n) \in \Lambda_0$ *such that* $b^k_{\sigma(n)}/b^{k_1}_{\sigma(n)} \leq |\lambda_n|$ *for all* n*.*

Remark. The criterion in 4.3 can be used to decide easily whether a non-archimedean countably normed Fréchet space with a Schauder basis is Λ_0-nuclear (recall that a such space can be identified with some $K(B)$).

Observe that since $\Lambda_0 \subset c_0$, every Λ_0-nuclear space is nuclear. But the converse is not true in general. Indeed, we know (see [7], Lemma 2.3) that Λ (or Λ_0) is nuclear if and only if there exists $\alpha \in P$ with $\alpha_n \to \infty$. However, we have the following

Proposition 4.4. *None of the spaces* Λ *and* Λ_0 *is* Λ_0*-nuclear.*

Proof. Suppose that one of the spaces Λ or Λ_0 is Λ_0-nuclear. By Theorem 4.2, given $\alpha \in P$, there exist $\beta \in P$ with $\alpha \ll \beta$, a permutation σ on N, and $(\lambda_n) \in \Lambda_0$ such that $\alpha_{\sigma(n)} \leq |\lambda_n| \beta_{\sigma(n)}$ for all n. It is easy to see that the set $N_1 = \{n \in N : n \geq \sigma(n)\}$ is infinite. For $n \in N_1$ we have

$$\alpha_1 \leq \alpha_{\sigma(n)} \leq |\lambda_n| \beta_{\sigma(n)} \leq |\lambda_n| \beta_n.$$

This contradicts the fact that $(\lambda_n) \in \Lambda_0$. □

Observe that every Λ_0-nuclear space has the Λ_0-property (Theorem 3.5). But the converse is not true in general. Indeed, if P is countable and K is not spherically complete, then $\bar{\Lambda}$ has the Λ_0-property (see the examples in Section 2). However, with regard to the Λ_0-nuclearity of $\bar{\Lambda}$, we have

Proposition 4.5. $\bar{\Lambda}$ *is* Λ_0*-nuclear if and only if* $\Lambda = \Lambda_0$*.*

Proof. Assume that $\Lambda = \Lambda_0$. Let $\xi = (\xi_n) \in \Lambda$ and let $\lambda \in K$ with $|\lambda| > 1$. For each $n \in N$, choose $\lambda_n \in K$ with $|\lambda_n| \leq \sqrt{|\xi_n|} \leq |\lambda\lambda_n|$. Then $(\lambda_n) \in \Lambda_0$ and $|\xi_n| \leq |\lambda_n|.|\lambda^2\lambda_n|$ for all n. By Theorem 4.2 we conclude that $\bar{\Lambda}$ is Λ_0-nuclear.

Conversely, assume that $\bar{\Lambda}$ is Λ_0-nuclear and let $\xi \in \Lambda$. By Theorem 4.2, there exist $y \in \Lambda$, a permutation σ on N, and $(\lambda_n) \in \Lambda_0$ such that $|\xi_{\sigma(n)}| \leq |\lambda_n y_{\sigma(n)}|$ for all n. Since $\lambda_n \to 0$, given $\epsilon > 0$ and $\alpha \in P$, there exits $m \in N$ such that $|\lambda_n| p_\alpha(y) < \epsilon$ if $n \geq m$. Then, for $n \geq m$ we have

$$\alpha_{\sigma(n)}|\xi_{\sigma(n)}| \leq |\lambda_n|\alpha_{\sigma(n)}|y_{\sigma(n)}| \leq |\lambda_n| p_\alpha(y) < \epsilon.$$

Hence, $\xi \in \Lambda_0$. □

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