

**ON THE NUMBER OF REPRESENTATIONS OF POSITIVE  
INTEGERS BY A DIRECT SUM OF BINARY  
QUADRATIC FORMS WITH DISCRIMINANT  $-23$**

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ABSTRACT. Explicit exact formulas are obtained for the number of representations of positive integers by some direct sums of quadratic forms  $F_1 = x_1^2 + x_1x_2 + 6x_2^2$  and  $\Phi_1 = 2x_1^2 + x_1x_2 + 3x_2^2$ .

Let

$$F_1 = x_1^2 + x_1x_2 + 6x_2^2 \quad \text{and} \quad \Phi_1 = 2x_1^2 + x_1x_2 + 3x_2^2.$$

These are primitive reduced binary quadratic forms with discriminant  $\Delta = -23$ . For each  $k \geq 1$ , let  $F_k$  and  $\Phi_k$  denote the direct sums of  $k$  copies of  $F_1$  and  $\Phi_1$ , respectively.

In [1] exact formulas are derived for the number of representations of a positive integer by positive quadratic forms in six variables with integral coefficients, among which there are suitable formulas for the quadratic forms  $F_3, F_2 \oplus \Phi_1, F_1 \oplus \Phi_2$  and  $\Phi_3$ .

In [2] explicit exact formulas are derived for the arithmetical function  $r(n; Q)$ , the number of representations of a positive integer  $n$  by the quadratic forms  $Q$ , for  $Q = F_2, F_1 \oplus \Phi_1$  and  $\Phi_2$ .

In the present paper we obtain formulas for  $r(n; Q)$  when  $Q = F_4, F_3 \oplus \Phi_1, F_2 \oplus \Phi_2, F_1 \oplus \Phi_3$  and  $\Phi_4$ .

It should be pointed out that the approach described here enables one to get formulas for  $r(n; Q)$  for  $Q = F_k, \Phi_k, F_i \oplus \Phi_j$  ( $i, j \geq 1, i + j = k$ ), where  $k > 4$ . However the calculations will be very tedious.

In this paper the notation, definitions, and some results from [3] will be mostly used.

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## § 1. SOME KNOWN RESULTS

Let

$$Q = Q(x_1, x_2, \dots, x_f) = \sum_{1 \leq r \leq s \leq f} b_{rs} x_r x_s$$

be a positive quadratic form in  $f$  ( $f$  is even) variables with integral coefficients  $b_{rs}$ . Further let  $\mathcal{D}$  be the determinant of the quadratic form

$$2Q = \sum_{r,s=1}^f a_{rs} x_r x_s \quad (a_{rr} = 2b_{rr}, \quad a_{rs} = a_{sr} = b_{rs}, \quad r < s);$$

$A_{rs}$  the algebraic cofactors of elements  $a_{rs}$  in  $\mathcal{D}$ ;  $\Delta$  the discriminant of the form  $Q$ , i.e.,  $\Delta = (-1)^{f/2} \mathcal{D}$ ;  $\delta = \text{g.c.d.}(\frac{A_{rr}}{2}, A_{rs})$  ( $r, s = 1, \dots, f$ );  $N = \frac{\mathcal{D}}{\delta}$  the level of the form  $Q$ ;  $\chi(d)$  the character of the form  $Q$ , i.e.,  $\chi(d) = 1$  if  $\Delta$  is a perfect square, but if  $\Delta$  is not a perfect square and  $2 \nmid \Delta$ , then  $\chi(d) = (\frac{d}{|\Delta|})$  for  $d > 0$  and  $\chi(d) = (-1)^{f/2} \chi(-d)$  for  $d < 0$  (here  $(\frac{d}{|\Delta|})$  is the generalized Jacobi symbol). A positive quadratic form in  $f$  variables of level  $N$  and character  $\chi(d)$  is called a quadratic form of type  $(-f/2, N, \chi)$ . Finally, let  $\mathcal{P}_\nu = \mathcal{P}_\nu(x_1, x_2, \dots, x_f)$  be a spherical function of order  $\nu$  ( $\nu$  is a positive integer) with respect to the quadratic form  $Q$ .

In what follows  $q$  is an odd prime and  $z = \exp(2\pi i\tau)$ ,  $\text{Im } \tau > 0$ .

As is wellknown, to each positive quadratic form  $Q$  there corresponds the theta-series

$$\vartheta(\tau; Q) = 1 + \sum_{n=1}^{\infty} r(n; Q) z^n. \quad (1.1)$$

We shall formulate the well-known results in the form of the following lemmas.

**Lemma 1** ([3], p. 874, 875, 817; see also [4], p. 15). *Any positive quadratic form  $Q$  of type  $(-k, q, 1)$ ,  $2|k$ ,  $k > 2$ , corresponds to one and the same Eisenstein series*

$$E(\tau; Q) = 1 + \sum_{n=1}^{\infty} (\alpha \sigma_{k-1}(n) z^n + \beta \sigma_{k-1}(n) z^{qn}),$$

where

$$\alpha = \frac{i^k}{\rho_k} \frac{q^{k/2} - i^k}{q^k - 1}, \quad \beta = \frac{1}{\rho_k} \frac{q^k - i^k q^{k/2}}{q^k - 1}, \quad (1.2)$$

$$\rho_k = (-1)^{k/2} \frac{(k-1)!}{(2\pi)^k} \zeta(k). \quad (1.3)$$

**Lemma 2** ([3], pp.874, 875, 895). *If  $Q$  is a primitive quadratic form of type  $(-k, q, 1)$ ,  $2|k$ , then the difference  $\vartheta(\tau; Q) - E(\tau; Q)$  is a cusp form of type  $(-k, q, 1)$ .*

**Lemma 3** ([3], p. 853, Theorem 33). *The homogeneous quadratic polynomials in  $f$  variables  $\varphi_{rs} = x_r x_s - \frac{1}{f} \frac{A_{rs}}{\mathcal{D}} 2Q$  ( $r, s = 1, 2, \dots, f$ ) are spherical functions of second order with respect to the positive quadratic form  $Q$  in  $f$  variables.*

**Lemma 4** ([3], p. 855). *If  $Q$  is a quadratic form of type  $(-f/2, N, \chi)$  and  $\mathcal{P}_\nu$  is the spherical function of order  $\nu$  with respect to  $Q$ , then the generalized multiple theta-series*

$$\vartheta(r; Q, \mathcal{P}_\nu) = \sum_{n=1}^{\infty} \left( \sum_{Q=n} \mathcal{P}_\nu \right) z^n$$

*is a cusp form of type  $(-(f/2 + \nu), N, \chi)$ .*

**Lemma 5** ([3], p.846). *If the quadratic forms  $Q_1$  and  $Q_2$  have the same level  $N$  and characters  $\chi_1(d)$  and  $\chi_2(d)$ , respectively, then the quadratic form  $Q_1 \oplus Q_2$  will have the level  $N$  and the character  $\chi_1(d)\chi_2(d)$ .*

## § 2. SOME AUXILIARY RESULTS

**2.1.** For the quadratic form  $F_1$  we have  $\mathcal{D} = 23$ ,  $A_{11} = 12$ ,  $A_{22} = 2$ , i.e.,  $\delta = 1$ ,  $N = 23$ ,  $\Delta = -23$ ,  $\chi = \chi(d) = \left(\frac{d}{23}\right)$  if  $d > 0$ . For the quadratic form  $\Phi_1$  we have  $\mathcal{D} = 23$ ,  $A_{11} = 6$ ,  $A_{22} = 4$ , i.e.,  $\delta = 1$ ,  $N = 23$ ,  $\Delta = -23$ ,  $\chi = \chi(d) = \left(\frac{d}{23}\right)$  if  $d > 0$ . Hence  $F_1$  and  $\Phi_1$  are quadratic forms of type  $(-1, 23, \chi)$ . Thus, by Lemma 5,  $F_2$ ,  $\Phi_2$ , and  $F_1 \oplus \Phi_1$  are quadratic forms of type  $(-2, 23, 1)$ .

For the quadratic form  $F_2$  we have  $\mathcal{D} = 23^2$ ,  $A_{11} = 12 \cdot 23$ ,  $A_{22} = 2 \cdot 23$ . Hence, if in Lemma 3 we put

$$f = 4, \quad Q = F_2, \quad r = s = 1,$$

then the polynomial

$$\varphi_{11} = x_1^2 - \frac{6}{23} F_2$$

will be a spherical function of second order with respect to  $F_2$ .

For the quadratic form  $\Phi_2$  we have  $\mathcal{D} = 23^2$ ,  $A_{11} = 6 \cdot 23$ ,  $A_{22} = 4 \cdot 23$ . Hence, if in Lemma 3 we put

$$f = 4, \quad Q = \Phi_2, \quad r = s = 1, \quad \text{and} \quad r = s = 2,$$

then the polynomials

$$\varphi_{11} = x_1^2 - \frac{3}{23} \Phi_2 \quad \text{and} \quad \varphi_{22} = x_2^2 - \frac{2}{23} \Phi_2$$

will be spherical functions of second order with respect to  $\Phi_2$ .

For the quadratic form  $F_1 \oplus \Phi_1$  we have  $\mathcal{D} = 23^2$ ,  $A_{11} = 12 \cdot 23$ ,  $A_{22} = 2 \cdot 23$ . Hence, if in Lemma 3 we put

$$f = 4, \quad Q = F_1 \oplus \Phi_1, \quad r = s = 1, \quad \text{and} \quad r = s = 2,$$

then the polynomials

$$\varphi_{11} = x_1^2 - \frac{6}{23}(F_1 \oplus \Phi_1) \quad \text{and} \quad \varphi_{22} = x_2^2 - \frac{1}{23}(F_1 \oplus \Phi_1)$$

will be spherical functions of second order with respect to  $F_1 \oplus \Phi_1$ .

**2.2.** It is easy to verify that the equation  $F_1 = n$

- (a) has two integral solutions for  $n = 1$ :  $x_1 = \pm 1$ ,  $x_2 = 0$ ;
- (b) has no integral solutions for  $n = 2, 3, 5$ ;
- (c) has two integral solutions for  $n = 4$ :  $x_1 = \pm 2$ ,  $x_2 = 0$ .

Also it is easy to verify that the equation  $\Phi_1 = n$

- (a) has no integral solutions for  $n = 1, 5$ ;
- (b) has two integral solutions for  $n = 2$ :  $x_1 = \pm 1$ ,  $x_2 = 0$ ;
- (c) has two integral solutions for  $n = 3$ :  $x_1 = 0$ ,  $x_2 = \pm 1$ .
- (d) has two integral solutions for  $n = 4$ :  $x_1 = \pm 1$ ,  $x_2 = \mp 1$ .

Hence, according to (1.1), we have

$$\vartheta(\tau; F_1) = 1 + 2z + 2z^4 + 0z^5 + \dots, \quad (2.1)$$

$$\vartheta(\tau; \Phi_1) = 1 + 2z^2 + 2z^3 + 2z^4 + 0z^5 + \dots. \quad (2.2)$$

From (2.1) it follows that

$$\vartheta(\tau; F_2) = \vartheta^2(\tau; F_1) = 1 + 4z + 4z^2 + 4z^4 + 8z^5 + \dots, \quad (2.3)$$

whence

$$\vartheta(\tau; F_3) = \vartheta(\tau; F_2)\vartheta(\tau; F_1) = 1 + 6z + 12z^2 + 8z^3 + 6z^4 + 24z^5 + \dots. \quad (2.4)$$

From (2.2) it follows that

$$\vartheta(\tau; \Phi_2) = \vartheta^2(\tau; \Phi_1) = 1 + 4z^2 + 4z^3 + 8z^4 + 8z^5 + \dots, \quad (2.5)$$

whence

$$\vartheta(\tau; \Phi_3) = \vartheta(\tau; \Phi_2)\vartheta(\tau; \Phi_1) = 1 + 6z^2 + 6z^3 + 18z^4 + 24z^5 + \dots. \quad (2.6)$$

From (2.1) and (2.2) it follows that

$$\begin{aligned} \vartheta(\tau; F_1 \oplus \Phi_1) &= \vartheta(\tau; F_1)\vartheta(\tau; \Phi_1) = \\ &= 1 + 2z + 2z^2 + 6z^3 + 8z^4 + 4z^5 + \dots. \end{aligned} \quad (2.7)$$

§ 3. FORMULAS FOR  $r(n; F_4)$ ,  $r(n; \Phi_4)$ ,  $r(n; F_3 \oplus \Phi_1)$ ,  $r(n; F_2 \oplus \Phi_2)$ ,  
 $r(n; F_1 \oplus \Phi_3)$

**Theorem 1.** *The system of generalized fourfold theta-series*

$$\vartheta(\tau; F_2, \varphi_{11}) = \frac{1}{23} \sum_{n=1}^{\infty} \left( \sum_{F_2=n} 23x_1^2 - 6n \right) z^n, \tag{3.1}$$

$$\vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{11}) = \frac{1}{23} \sum_{n=1}^{\infty} \left( \sum_{F_1 \oplus \Phi_1=n} 23x_1^2 - 6n \right) z^n, \tag{3.2}$$

$$\vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{22}) = \frac{1}{23} \sum_{n=1}^{\infty} \left( \sum_{F_1 \oplus \Phi_1=n} 23x_2^2 - n \right) z^n, \tag{3.3}$$

$$\vartheta(\tau; \Phi_2, \varphi_{11}) = \frac{1}{23} \sum_{n=1}^{\infty} \left( \sum_{\Phi_2=n} 23x_1^2 - 3n \right) z^n, \tag{3.4}$$

$$\vartheta(\tau; \Phi_2, \varphi_{22}) = \frac{1}{23} \sum_{n=1}^{\infty} \left( \sum_{\Phi_2=n} 23x_2^2 - 2n \right) z^n \tag{3.5}$$

is a basis of the space  $S_4(23, 1)$  (the space of cusp forms of type  $(-4, 23, 1)$ ).

*Proof.* I. As said above,  $F_2$  is a quadratic form of type  $(-2, 23, 1)$  and

$$\varphi_{11} = x_1^2 - \frac{6}{23} F_2$$

is a spherical function of second order with respect to  $F_2$ . Hence, by Lemma 4, the theta-series (3.1) is a cusp form of type  $(-4, 23, 1)$ .

Taking into account (2.3), it is not difficult to verify that the equation  $F_2 = n$

- (a) has four integral solutions for  $n = 1$ :  $x_1 = \pm 1, x_2 = x_3 = x_4 = 0$ ;  
 $x_3 = \pm 1, x_1 = x_2 = x_4 = 0$ ;
- (b) has four integral solutions for  $n = 3$ :  $x_1 = \pm 1, x_3 = 1, x_2 = x_4 = 0$ ;  
 $x_1 = \pm 1, x_3 = -1, x_2 = x_4 = 0$  for  $n = 2$ ;
- (c) has no integral solutions for  $n = 3$ ;
- (d) has four integral solutions for  $n = 4$ :  $x_1 = \pm 2, x_2 = x_3 = x_4 = 0$ ;  
 $x_3 = \pm 2, x_1 = x_2 = x_4 = 0$ ;
- (e) has eight integral solutions for  $n = 5$ :  $x_1 = \pm 2, x_3 = 1, x_2 = x_4 = 0$ ;  
 $x_1 = \pm 2, x_3 = -1, x_2 = x_4 = 0$ ;  $x_1 = \pm 1, x_3 = 2, x_2 = x_4 = 0$ ;  
 $x_1 = \pm 1, x_3 = -2, x_2 = x_4 = 0$ .

Hence

$$\begin{aligned} \vartheta(\tau; F_2, \varphi_{11}) &= \frac{1}{23} \{ (23 \cdot 2 - 6 \cdot 4)z + (23 - 6 \cdot 2)4z^2 + \\ &+ (23 \cdot 4 \cdot 2 - 6 \cdot 4 \cdot 4)z^4 + (23 \cdot 4 \cdot 4 + 23 \cdot 4 - 6 \cdot 5 \cdot 8)z^5 + \dots \} = \end{aligned}$$

$$= \frac{22}{23}z + \frac{44}{23}z^2 + \frac{88}{23}z^4 + \frac{220}{23}z^5 + \dots \quad (3.6)$$

II. As said above  $F_1 \oplus \Phi_1$  is a quadratic form of type  $(-2, 23, 1)$  and

$$\varphi_{11} = x_1^2 - \frac{6}{23}(F_1 \oplus \Phi_1) \quad \text{and} \quad \varphi_{22} = x_2^2 - \frac{1}{23}(F_1 \oplus \Phi_1)$$

are spherical functions of second order with respect to  $F_1 \oplus \Phi_1$ .

Hence, by Lemma 4, the theta-series (3.2) and (3.3) are cusp forms of type  $(-4, 23, 1)$ .

Taking into account (2.7), it is not difficult to verify that the equation  $F_1 \oplus \Phi_1 = n$

- (a) has two integral solutions for  $n = 1$ :  $x_1 = \pm 1, x_2 = x_3 = x_4 = 0$ ;
- (b) has two integral solutions for  $n = 2$ :  $x_3 = \pm 1, x_1 = x_2 = x_4 = 0$ ;
- (c) has six integral solutions for  $n = 3$ :  $x_1 = \pm 1, x_3 = 1, x_2 = x_4 = 0$ ;  
 $x_1 = \pm 1, x_3 = -1, x_2 = x_4 = 0$ ;  $x_1 = x_2 = x_3 = 0, x_4 = \pm 1$ ;
- (d) has eight integral solutions for  $n = 4$ :  $x_1 = \pm 1, x_4 = 1, x_2 = x_3 = 0$ ;  
 $x_1 = \pm 1, x_4 = -1, x_2 = x_3 = 0$ ;  $x_1 = \pm 2, x_2 = x_3 = x_4 = 0$ ;  
 $x_1 = x_2 = 0, x_3 = 1, x_4 = -1$ ;  $x_1 = x_2 = 0, x_3 = -1, x_4 = 1$ ;
- (e) has four integral solutions for  $n = 5$ :  $x_1 = \pm 1, x_3 = 1, x_4 = -1,$   
 $x_2 = 0$ ;  $x_1 = \pm 1, x_2 = 0, x_3 = -1, x_4 = 1$ .

Hence,

$$\begin{aligned} \vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{11}) &= \frac{1}{23} \{ (23-6)2z + (-6 \cdot 2 \cdot 2)z^2 + \\ &+ (23 \cdot 4 - 6 \cdot 3 \cdot 6)z^3 + (23 \cdot 4 \cdot 2 + 23 \cdot 4 - 6 \cdot 4 \cdot 8)z^4 + (23 - 6 \cdot 5 \cdot 4)z^5 + \dots \} = \\ &= \frac{34}{23}z - \frac{24}{23}z^2 - \frac{16}{23}z^3 + \frac{84}{23}z^4 - \frac{28}{23}z^5 + \dots, \end{aligned} \quad (3.7)$$

$$\begin{aligned} \vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{22}) &= \\ &= \frac{1}{23} \{ -2z - 2 \cdot 2z^2 - 3 \cdot 6z^3 - 4 \cdot 8z^4 - 5 \cdot 4z^5 + \dots \} = \\ &= -\frac{2}{23}z - \frac{4}{23}z^2 - \frac{18}{23}z^3 - \frac{32}{23}z^4 - \frac{20}{23}z^5 + \dots \end{aligned} \quad (3.8)$$

III. As said above  $\Phi_2$  is a quadratic form of type  $(-2, 23, 1)$  and

$$\varphi_{11} = x_1^2 - \frac{3}{23}\Phi_2 \quad \text{and} \quad \varphi_{22} = x_2^2 - \frac{2}{23}\Phi_2$$

are spherical functions of second order with respect to  $\Phi_2$ . Hence, by Lemma 4, theta-series (3.4) and (3.5) are cusp forms of type  $(-4, 23, 1)$ .

Taking into account (2.5) it is easy to verify that the equation  $\Phi_2 = n$

- (a) has no integral solutions for  $n = 1$ ;
- (b) has four integral solutions for  $n = 2$ :  $x_1 = \pm 1, x_2 = x_3 = x_4 = 0$ ;  
 $x_3 = \pm 1, x_1 = x_2 = x_4 = 0$ ;

- (c) has four integral solutions for  $n = 3$ :  $x_2 = \pm 1$ ,  $x_1 = x_3 = x_4 = 0$ ;  
 $x_1 = x_2 = x_3 = 0$ ,  $x_4 = \pm 1$ ;
- (d) has eight integral solutions for  $n = 4$ :  $x_1 = \pm 1$ ,  $x_3 = 1$ ,  $x_2 = x_4 = 0$ ;  
 $x_1 = \pm 1$ ,  $x_3 = -1$ ,  $x_2 = x_4 = 0$ ;  $x_1 = 1$ ,  $x_2 = -1$ ,  $x_3 = x_4 = 0$ ;  
 $x_1 = -1$ ,  $x_2 = 1$ ,  $x_3 = x_4 = 0$ ;  $x_1 = x_2 = 0$ ,  $x_3 = 1$ ,  $x_4 = -1$ ;  
 $x_1 = x_2 = 0$ ,  $x_3 = -1$ ,  $x_4 = 1$ ;
- (e) has eight integral solutions for  $n = 5$ :  $x_1 = \pm 1$ ,  $x_2 = x_3 = 0$ ,  
 $x_4 = 1$ ;  $x_1 = \pm 1$ ,  $x_2 = x_3 = 0$ ,  $x_4 = -1$ ;  $x_1 = x_4 = 0$ ,  $x_2 = \pm 1$ ,  
 $x_3 = 1$ ;  $x_1 = x_4 = 0$ ,  $x_2 = \pm 1$ ,  $x_3 = -1$ .

Hence

$$\begin{aligned} \vartheta(\tau; \Phi_2, \varphi_{11}) &= \frac{1}{23} \{ (23 \cdot 1 \cdot 2 - 3 \cdot 2 \cdot 4)z^2 - 3 \cdot 3 \cdot 4z^3 + \\ &+ (23 \cdot 1 \cdot 6 - 3 \cdot 4 \cdot 8)z^4 + (23 \cdot 1 \cdot 4 - 3 \cdot 5 \cdot 8)z^5 + \dots \} = \\ &= \frac{22}{23} z^2 - \frac{36}{23} z^3 + \frac{42}{23} z^4 - \frac{28}{23} z^5 + \dots, \end{aligned} \quad (3.9)$$

$$\begin{aligned} \vartheta(\tau; \Phi_2, \varphi_{22}) &= \frac{1}{23} \{ -2 \cdot 2 \cdot 4z^2 + (23 \cdot 1 \cdot 2 - 2 \cdot 3 \cdot 4)z^3 + \\ &+ (23 \cdot 1 \cdot 2 - 2 \cdot 4 \cdot 8)z^4 + (23 \cdot 1 \cdot 4 - 2 \cdot 5 \cdot 8)z^5 + \dots \} = \\ &= -\frac{16}{23} z^2 + \frac{22}{23} z^3 - \frac{18}{23} z^4 + \frac{12}{23} z^5 + \dots. \end{aligned} \quad (3.10)$$

The system of theta-series (3.1)–(3.5) is linearly independent, since the determinant of fifth order whose elements are coefficients in the expansions of these theta-series is different from zero. Thus the theorem is proved, since  $\dim S_4(23, 1) = 5$  [3, p. 900].  $\square$

**Theorem 2.**

$$\begin{aligned} r(n; F_4) &= \frac{24}{53} \sigma_3^*(n) - \frac{240}{11 \cdot 53} \left( \sum_{F_2=n} 23x_1^2 - 6n \right) + \\ &+ \frac{440}{23 \cdot 53} \left( \sum_{F_1 \oplus \Phi_1=n} 23x_1^2 - 6n \right) - \frac{2640}{23 \cdot 53} \left( \sum_{F_1 \oplus \Phi_1=n} 23x_2^2 - n \right) - \\ &- \frac{8144}{23 \cdot 53} \left( \sum_{\Phi_2=n} 23x_1^2 - 3n \right) - \frac{14096}{23 \cdot 53} \left( \sum_{\Phi_2=n} 23x_2^2 - 2n \right), \end{aligned} \quad (I)$$

$$\begin{aligned} r(n; \Phi_4) &= \frac{24}{53} \sigma_3^*(n) - \frac{28}{11 \cdot 53} \left( \sum_{F_2=n} 23x_1^2 - 6n \right) + \\ &+ \frac{16}{23 \cdot 53} \left( \sum_{F_1 \oplus \Phi_1=n} 23x_1^2 - 6n \right) - \frac{96}{23 \cdot 53} \left( \sum_{F_1 \oplus \Phi_1=n} 23x_2^2 - n \right) - \end{aligned}$$

$$-\frac{512}{23 \cdot 53} \left( \sum_{\Phi_2=n} 23x_1^2 - 3n \right) - \frac{1164}{23 \cdot 53} \left( \sum_{\Phi_2=n} 23x_2^2 - 2n \right), \quad (\text{II})$$

$$\begin{aligned} r(n; F_3 \oplus \Phi_1) &= \frac{24}{53} \sigma_3^*(n) - \frac{28}{11 \cdot 53} \left( \sum_{F_2=n} 23x_1^2 - 6n \right) + \\ &+ \frac{175}{23 \cdot 53} \left( \sum_{F_1 \oplus \Phi_1=n} 23x_1^2 - 6n \right) - \frac{1050}{23 \cdot 53} \left( \sum_{F_1 \oplus \Phi_1=n} 23x_2^2 - n \right) - \\ &- \frac{2685}{23 \cdot 53} \left( \sum_{\Phi_2=n} 23x_1^2 - 3n \right) - \frac{4609}{23 \cdot 53} \left( \sum_{\Phi_2=n} 23x_2^2 - 2n \right), \quad (\text{III}) \end{aligned}$$

$$\begin{aligned} r(n; F_2 \oplus \Phi_2) &= \frac{24}{53} \sigma_3^*(n) - \frac{28}{11 \cdot 53} \left( \sum_{F_2=n} 23x_1^2 - 6n \right) + \\ &+ \frac{122}{23 \cdot 53} \left( \sum_{F_1 \oplus \Phi_1=n} 23x_1^2 - 6n \right) - \frac{732}{23 \cdot 53} \left( \sum_{F_1 \oplus \Phi_1=n} 23x_2^2 - n \right) - \\ &- \frac{1360}{23 \cdot 53} \left( \sum_{\Phi_2=n} 23x_1^2 - 3n \right) - \frac{2330}{23 \cdot 53} \left( \sum_{\Phi_2=n} 23x_2^2 - 2n \right), \quad (\text{IV}) \end{aligned}$$

$$\begin{aligned} r(n; F_1 \oplus \Phi_3) &= \frac{24}{53} \sigma_3^*(n) - \frac{28}{11 \cdot 53} \left( \sum_{F_2=n} 23x_1^2 - 6n \right) + \\ &+ \frac{3}{53} \left( \sum_{F_1 \oplus \Phi_1=n} 23x_1^2 - 6n \right) - \frac{18}{53} \left( \sum_{F_1 \oplus \Phi_1=n} 23x_2^2 - n \right) - \\ &- \frac{1201}{23 \cdot 53} \left( \sum_{\Phi_2=n} 23x_1^2 - 3n \right) - \frac{1959}{23 \cdot 53} \left( \sum_{\Phi_2=n} 23x_2^2 - 2n \right), \quad (\text{V}) \end{aligned}$$

where

$$\begin{aligned} \sigma_3^*(n) &= \sigma_3(n) && \text{if } 23 \nmid n, \\ &= \sigma_3(n) + 23^2 \sigma_3\left(\frac{n}{23}\right) && \text{if } 23 | n. \end{aligned}$$

*Proof.* From (2.3), (2.5) follow the relations

$$\vartheta(\tau; F_4) = \vartheta^2(\tau; F_2) = 1 + 8z + 24z^2 + 32z^3 + 24z^4 + 48z^5 + \dots, \quad (3.11)$$

$$\vartheta(\tau; \Phi_4) = \vartheta^2(\tau; \Phi_2) = 1 + 8z^2 + 8z^3 + 32z^4 + 48z^5 + \dots, \quad (3.12)$$

respectively. The couples of relations (2.2) and (2.4), (2.3) and (2.4), (2.1) and (2.6) lead to

$$\begin{aligned} \vartheta(\tau; F_3 \oplus \Phi_1) &= \vartheta(\tau; F_3) \vartheta(\tau; \Phi_1) = \\ &= 1 + 6z + 14z^2 + 22z^3 + 44z^4 + 76z^5 + \dots, \quad (3.13) \end{aligned}$$



$$\begin{aligned} \vartheta(\tau; F_2 \oplus \Phi_2) &= \vartheta(\tau; F_2)\vartheta(\tau; \Phi_2) = \\ &= 1 + 4z + 8z^2 + 20z^3 + 44z^4 + 64z^5 + \dots, \end{aligned} \quad (3.14)$$

$$\begin{aligned} \vartheta(\tau; F_1 \oplus \Phi_3) &= \vartheta(\tau; F_1)\vartheta(\tau; \Phi_3) = \\ &= 1 + 2z + 6z^2 + 18z^3 + 32z^4 + 60z^5 + \dots, \end{aligned} \quad (3.15)$$

respectively. By Lemma 5,  $F_4$ ,  $\Phi_4$ ,  $F_3 \oplus \Phi_1$ ,  $F_2 \oplus \Phi_2$  and  $F_1 \oplus \Phi_3$  are quadratic forms of type  $(-4, 23, 1)$ , to which by Lemma 1 there corresponds one and the same Eisenstein series. For  $k = 4$ , from (1.2) we have

$$\alpha = \frac{1}{\rho_4} \frac{23^2 - 1}{23^4 - 1} = \frac{1}{\rho_4} \frac{1}{23^2 + 1}, \quad \beta = \frac{1}{\rho_4} \frac{23^4 - 23^2}{23^4 - 1} = \frac{1}{\rho_4} \frac{23^2}{23^2 + 1}$$

where  $\rho_4 = \frac{1}{240}$  [3, p. 823]. Thus for all these forms we have

$$\begin{aligned} E(\tau; F_4) &= E(\tau; \Phi_4) = E(\tau; F_3 \oplus \Phi_1) = E(\tau; F_2 \oplus \Phi_2) = E(\tau; F_1 \oplus \Phi_3) = \\ &= 1 + \frac{24}{53} \sum_{n=1}^{\infty} (\sigma_3(n)z^n + 23^2\sigma_3(n)z^{23n}) = \end{aligned} \quad (3.16)$$

$$= 1 + \frac{24}{53}z + \frac{24 \cdot 9}{53}z^2 + \frac{24 \cdot 28}{53}z^3 + \frac{24 \cdot 73}{53}z^4 + \frac{24 \cdot 126}{53}z^5 + \dots \quad (3.16_1)$$

(I). By Lemma 2, the difference  $\vartheta(\tau; F_4) - E(\tau; F_4)$  is a cusp form of type  $(-4, 23, 1)$ . Hence, by Theorem 1, there exist numbers  $c_1, \dots, c_5$  such that

$$\begin{aligned} \vartheta(\tau; F_4) - E(\tau; F_4) &= c_1\vartheta(\tau; F_2, \varphi_{11}) + c_2\vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{11}) + \\ &+ c_3\vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{22}) + c_4\vartheta(\tau; \Phi_2, \varphi_{11}) + c_5\vartheta(\tau; \Phi_2, \varphi_{22}). \end{aligned}$$

Equating the coefficients of  $z, z^2, \dots, z^5$  on both sides of this equality and taking into account (3.11), (3.16<sub>1</sub>) and (3.6)–(3.10), we can find these numbers and obtain

$$\begin{aligned} \vartheta(\tau, F_4) &= E(\tau; F_4) - \frac{240 \cdot 23}{11 \cdot 53} \vartheta(\tau; F_2, \varphi_{11}) + \\ &+ \frac{440}{53} \vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{11}) - \frac{2640}{53} \vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{22}) - \\ &- \frac{8144}{53} \vartheta(\tau; \Phi_2, \varphi_{11}) - \frac{14096}{53} \vartheta(\tau; \Phi_2, \varphi_{22}). \end{aligned}$$

Equating now the coefficients of  $z^n$  on both sides of this equation, by (1.1), (3.16), and (3.1)–(3.5), we get the desired formula (I).

(II)–(V). Applying the same arguments as above except that (3.11) is replaced by (3.12), (3.13), (3.14), and (3.15) for the quadratic forms  $\Phi_4$ ,  $\Phi_3 \oplus \Phi_1$ ,  $F_2 \oplus \Phi_2$ , and  $\Phi_1 \oplus \Phi_3$ , respectively, we obtain

$$\vartheta(\tau, \Phi_4) = E(\tau; \Phi_4) - \frac{644}{11 \cdot 53} \vartheta(\tau; F_2, \varphi_{11}) +$$

$$\begin{aligned}
& + \frac{16}{53} \vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{11}) - \frac{96}{53} \vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{22}) - \\
& - \frac{512}{53} \vartheta(\tau; \Phi_2, \varphi_{11}) - \frac{1164}{53} \vartheta(\tau; \Phi_2, \varphi_{22}), \\
\vartheta(\tau, F_3 \oplus \Phi_1) & = E(\tau; F_3 \oplus \Phi_1) - \frac{644}{11 \cdot 53} \vartheta(\tau; F_2, \varphi_{11}) + \\
& + \frac{175}{53} \vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{11}) - \frac{1050}{53} \vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{22}) - \\
& - \frac{2685}{53} \vartheta(\tau; \Phi_2, \varphi_{11}) - \frac{4609}{53} \vartheta(\tau; \Phi_2, \varphi_{22}), \\
\vartheta(\tau, F_2 \oplus \Phi_2) & = E(\tau; F_2 \oplus \Phi_2) - \frac{644}{11 \cdot 53} \vartheta(\tau; F_2, \varphi_{11}) + \\
& + \frac{122}{53} \vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{11}) - \frac{732}{53} \vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{22}) - \\
& - \frac{1360}{53} \vartheta(\tau; \Phi_2, \varphi_{11}) - \frac{2330}{53} \vartheta(\tau; \Phi_2, \varphi_{22}), \\
\vartheta(\tau, F_1 \oplus \Phi_3) & = E(\tau; F_1 \oplus \Phi_3) - \frac{644}{11 \cdot 53} \vartheta(\tau; F_2, \varphi_{11}) + \\
& + \frac{69}{53} \vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{11}) - \frac{414}{53} \vartheta(\tau; F_1 \oplus \Phi_1, \varphi_{22}) - \\
& - \frac{1201}{53} \vartheta(\tau; \Phi_2, \varphi_{11}) - \frac{1959}{53} \vartheta(\tau; \Phi_2, \varphi_{22}).
\end{aligned}$$

From these identities, as above, we get the formulas (II)–(V).  $\square$

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